Nonreductive WZW models and their CFTs, II: N=1 and N=2 cosets

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Abstract

In [6] we started a programme devoted to the systematic study of the conformal field theories derived from WZW models based on nonreductive Lie groups. In this, the second part, we continue this programme with a look at the N=1 and N=2 superconformal field theories which arise from both gauged and ungauged supersymmetric WZW models. We extend the supersymmetric (affine) Sugawara and coset constructions, as well as the Kazama–Suzuki construction to general self-dual Lie algebras.

1 Introduction

One of the most important problems in string theory is the construction of exact (super)string backgrounds. As a string propagates in a manifold M, it interacts with it via the following geometric data: a metric g, an antisymmetric two-form b, and a scalar field φ. Consistency imposes severe restrictions on the *background* (M, g, b, φ), equivalent to demanding the exact conformal invariance of the effective two-dimensional quantum field theory on the string world-sheet. For a general string background (M, g, b, φ), this translates into complicated coupled nonlinear partial differential equations for g, b, and φ involving an infinite number of perturbative α' corrections: the vanishing of the exact β-function. The exact form of the equations is therefore not known,

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and although presumably starting from any classical background one can flow under the renormalisation group to a nearby exact background, it is a fact that very few exact string backgrounds are known explicitly.

One way to construct exact string backgrounds is to start with an exact string background and perform operations which preserve both the conformal invariance and the spacetime interpretation of string propagation. For example, if we start with a string propagating on a flat Minkowski spacetime, one can, via toroidal compactifications and orbifold constructions, reach other more realistic exact string backgrounds. More generally, one can start with string propagating on a group manifold. For string propagation to be consistent, however, the group must possess a bi-invariant metric; that is, a metric invariant under both left and right multiplications. Equivalently its Lie algebra must be self-dual (see, for example, [7]). Every compact Lie group has a self-dual Lie algebra, as can be easily shown by averaging over the group using the Haar measure, but these are not the only Lie groups with this property. The conformal field theory describing string propagation on a group manifold is the WZW model. The conformal invariance of the WZW model derives, via the affine Sugawara construction, from its huge semi-local symmetry. Gauging some of this symmetry one can then obtain other exact string backgrounds, some of which possess a spacetime interpretation as string propagation on coset spaces.

Until relatively recently, most of the work on the construction of string backgrounds starting from WZW models (or their supersymmetric generalisations) was concerned with compact (or more generally, reductive) Lie groups. Reductive Lie groups are (up to coverings) direct product of semisimple and abelian Lie groups, hence this class of string backgrounds also comprises the flat spacetimes and the toroidal compactifications. But ever since the work of Nappi & Witten [19], in which an exact four-dimensional string background describing a gravitational wave was obtained from the WZW model corresponding to a nonsemisimple (in fact, solvable) Lie group, the possibility has arisen of considering more general Lie groups. In a recent paper [6], of which the present paper is a continuation and will hereafter be referred to as Part I, we have analysed in detail the construction, conformal invariance and gauging of WZW models based on nonreductive Lie groups, as well as the conformal field theories they give rise to. We refer the reader to this paper for references to the work on nonreductive WZW models.

In the present paper we extend the results of Part I to supersymmetric WZW models and their $N{=}1$ and $N{=}2$ supersymmetric cosets. Whereas the relation between gauged WZW models and coset constructions was already well-understood in the reductive case (see Part I and references therein), this happy state of affairs did not persist in the supersymmetric situation. Two coset constructions are known which yield superconformal field theories: the one by

Goddard, Kent & Olive (GKO) in [10] and the one by Kazama & Suzuki (KS) in [14] (see also [25]).

The GKO construction is essentially a coset construction [2] of the form $\mathfrak{g}\times\mathfrak{g}/\mathfrak{g}$ where the first $\mathfrak{g}$ in the numerator is realised as a WZW model and the second $\mathfrak{g}$ as free fermions in the adjoint representation; that is, the numerator corresponds to a supersymmetric WZW model. This theory is superconformal, but the superconformal symmetry found by GKO after quotienting seems accidental; that is, it is not related to the superconformal symmetry of the unquotiented theory. A path integral derivation of the GKO construction, starting from a SWZW model, was given in [24] (see also [18]). This construction starts by gauging a bosonic diagonal subalgebra $\mathfrak{h}$ in the SWZW model. A superconformal theory is recovered only when $\mathfrak{h} = \mathfrak{g}$, precisely as in the GKO construction. With hindsight this result is to be expected: supersymmetry demands a delicate balance between the fermionic and bosonic degrees of freedom, which is upset if one gauges a symmetry which only gets rid of, say, bosonic degrees of freedom. What is surprising about the results of [10] and [24] is that the full diagonal gauging should give rise to a superconformal theory at all.

On the other hand, the KS construction is a natural superconformal coset where the superconformal symmetry is preserved along the way. However a satisfactory lagrangian description for the KS coset has taken longer to appear. In his work on topologically twisted KS cosets, Witten [29] wrote down without derivation a lagrangian for the coset theory. It was Tseytlin [27] (although see earlier work by Nojiri [20] in a particular case) who first wrote down a path-integral derivation of the KS construction for an arbitrary compact Lie group, starting from a SWZW model and gauging a diagonal subgroup of both bosonic and fermionic symmetries. Although the construction in [27] assumes that the Lie group be compact, a similar construction extends to the general case, as we will see below in more detail. These results notwithstanding, a conformal field-theoretical derivation of the KS coset constructions from a gauged WZW model, in the style of [13], did not exist even in the reductive case.

Even less is known in the nonreductive case. Indeed, at the time of writing the only work that had been done in this direction has been the generalisation of the N=1 Sugawara construction [16]. The original motivation of this paper was to fill this gap. Some of those results in this paper which transcend the nonreductive programme started in Part I have already been announced in our letter [8], where among other things we present for the first time (although see also [23]) a conformal field theoretical proof of the superconformal invariance of the gauged SWZW model.

Although the SWZW model with or without gauging is always invariant un-

der an N=1 superconformal algebra, it sometimes admits an extended N=2 superconformal symmetry. For arbitrary supersymmetric σ-models, Hull and Witten [11] wrote down the necessary conditions for the existence of a (linearly realised) such N=2 supersymmetry. When specialised to a SWZW model, these conditions translate into algebraic conditions on the Lie algebra, which were first obtained by the Leuven group [26]. These conditions were later re-interpreted in terms of Manin triples by Parkhomenko [21], and further elaborated by Getzler [9], who also discussed the coset construction. All these papers treat only the reductive case, but many of their results extend straightforwardly to the nonreductive case, as was shown by Mohammedi [17] and also by one of us [5]. In this latter paper the connection with Manin triples was re-established and a classification is given of those SWZW models based on a solvable Lie group which admit an N=2 superconformal symmetry with central charge c=9.

In the same way, under some conditions, the KS coset construction admits an N=2 superconformal symmetry; and in fact, with hindsight, the N=2 superconformal symmetry of the SWZW model (when it exists) is a special case of the N=2 superconformal invariance of the KS coset construction, where one gauges the trivial group. In this paper we will extend to the nonreductive case the N=2 KS coset construction, and in particular show how the extra symmetry comes induced from the gauged SWZW model.

We now come to a guided tour through the contents of this paper, but first a word on superspace versus components. Although in practice many practitioners in this field feel they have to choose between one convention or another, it is abundantly clear that each convention has its virtues as well as its shortcomings. As a result of this tendency, much of the literature contains partial results which have been obtained using one of the approaches, and the emerging picture—even in the reductive case—is not completely satisfying. Motivated by this, we will work both in superspace and in components, in an attempt to clarify at the same time the various connections between the two approaches in the context of the SWZW model.

Maybe the most natural way of defining the N=1 SWZW model is to covariantise the WZW model supersymmetrically. Formal similarity aside, the superspace formulation also turns out to yield the most direct and non-ambiguous proof of the superconformal invariance of the theory. We begin thus in Section 2 by describing the N=1 SWZW model based on a general self-dual Lie group in terms of superfields. We write down the action, examine its symmetries, determine its current algebra and prove its conformal invariance using the self-dual N=1 Sugawara construction.

We turn then our attention to the gauged SWZW model. In Section 3, still working in superfields, we gauge a diagonal subgroup of the isometries of the

model. Having done this, in Section 4 we prove the superconformal invariance of the model and determine exactly the corresponding SCFT. More precisely, we define the self-dual KS coset and we prove that this is precisely the superconformal field theory of the gauged SWZW model. This is done using the BRST formalism in superspace. The calculations rely on the superspace operator product expansion, and in the Appendix we write down the "superspace Borcherds axioms" for what we believe to be the first time (although see [15]).

The main appeal of the component formulation of the SWZW theory is its apparent simplicity. After eliminating the auxiliary fields, the SWZW action can be written as a WZW model coupled axially to Weyl fermions in the adjoint representation. For the gauged SWZW model, as remarked above, Witten [29] wrote down an action which describes the gauged WZW model coupled minimally to Weyl fermions on the coset directions. Its simplicity notwithstanding, this formulation still reserves us a few surprises.

An immediate choice which presents itself is whether to break up the action into components before or after gauging. The symmetries involved are sufficiently different that the equivalence of the two methods is not obvious. Therefore we treat both cases and show that they are equivalent and in fact equivalent to Witten's action.

In Section 5 we break the SWZW action into components. As we will see, the parametrisation of the superfield into bosonic and fermionic components is not unique. Different parametrisations are related by chiral gauge transformations, which although perfectly valid in the classical theory, are anomalous quantum mechanically. This means that different parametrisations yield different quantum theories. As we will see, the superconformal invariance of the quantum theory will give us the key to solve the ambiguity in the parametrisation of the superfield. Once having chosen a parametrisation, we eliminate the auxiliary fields, and determine its symmetries and the associated conserved currents.

Section 6 is dedicated to the gauged SWZW model and to the derivation of Witten's action. We start by gauging the component action, which as we will see, is not completely straightforward. We then take the gauged SWZW model in superfields and show that after eliminating the auxiliary fields, the resulting action is precisely Witten's action; and moreover that it agrees with he action obtained by gauging the component SWZW model.

In Section 7, we consider the self-dual Kazama–Suzuki cosets which possess an extended $N{=}2$ superconformal symmetry. For the results of this section it is necessary to work in components, since the expression for the generators of the extended supersymmetry is not local in the original superfields. In this section we derive the conditions under which a supersymmetric cosets admits $N{=}2$ supersymmetry and show that this symmetry is present already in the

gauged SWZW model description.

Let us close with a notational remark. We will be using freely the notation and results of [6] and any reference to an equation, section, theorem in that paper will be prefixed by an "I", so that, for example, equation (I.m.n) refers to equation in (m.n) in [6].

2 The N=1 SWZW model

In this section we introduce the N=1 supersymmetric WZW model associated with a Lie group possessing a bi-invariant metric. Starting with the action written in terms of superfields, we derive the classical and quantum algebras of currents, after which we review the N=1 Sugawara construction. This proves the superconformal invariance of the supersymmetric WZW model.

2.1 The N=1 SWZW model

We shall construct the N=1 supersymmetric WZW model (SWZW model, for short) as a supersymmetric covariantisation of the WZW model. The WZW model is manifestly a classical conformal field theory, and the fact that this persists at the quantum level can be attributed, via the Sugawara construction, to the affine Lie algebra of its conserved currents. Similarly, the SWZW model, when written in superfields, can be seen to be manifestly a classical superconformal field theory; and the N=1 Sugawara construction will guarantee that this continues to hold at the quantum level.

The data defining this model is a connected Lie group G possessing a bi-invariant metric; that is, a metric invariant under both right and left multiplication in G. This condition can also be understood in terms of the Lie algebra $\mathfrak{g}$. Any metric on $\mathfrak{g}$ can be automatically promoted to a left-invariant metric on G simply by identifying the Lie algebra with the left-invariant vector fields. A necessary and sufficient condition for this metric to be also right-invariant is that the metric on the Lie algebra be ad-invariant; that is, invariant under the infinitesimal adjoint action. Not every Lie algebra will possess such a metric: those which do are called *self-dual*. Semisimple and abelian Lie algebras comprise a small subclass of the self-dual Lie algebras, which themselves for a very small subclass of all Lie algebras. Although the structure of self-dual Lie algebras is mostly under control, a classification is still lacking. This is an interesting problem, for as shown in [16] (see also [4]), self-dual Lie algebras are in one-to-one correspondence with (S)WZW models. For a recent survey of results on self-dual Lie algebras see [7].

In this section we present the superfield description of the SWZW model. Our conventions for superspace and superspace operator product expansions are summarised in Appendix A, which the reader is invited to peruse at this stage. The fundamental fields will therefore be superfields $\mathbb{G}(Z,\bar{Z})$, where $Z=(z,\theta)$ and $\bar{Z}=(\bar{z},\bar{\theta})$ are coordinates in an N=1 super-Riemann surface Σ_S whose body is a Riemann surface Σ. The θ-independent component of $\mathbb{G}(Z,\bar{Z})$ is then a G-valued field $g(z,\bar{z})$ on Σ. Although the following nomenclature is strictly speaking not correct, we will speak of $\mathbb{G}$ as a G-valued superfield. We will have more to say about what $\mathbb{G}$ is later on in Section 5, when we discuss the component action.

Classically, the supersymmetric WZW model is defined by the following action, directly generalising the one in [1] for the reductive case:

$$I_\Omega[\mathbb{G}] = \int_{\Sigma_S} \left\langle \mathbb{G}^{-1} D\mathbb{G} \, , \mathbb{G}^{-1}\bar{D}\mathbb{G}\right\rangle + \int_{B_S} \left\langle \widetilde{\mathbb{G}}^{-1}\partial_t\widetilde{\mathbb{G}} \, , \left[\widetilde{\mathbb{G}}^{-1}D\widetilde{\mathbb{G}}, \widetilde{\mathbb{G}}^{-1}\bar{D}\widetilde{\mathbb{G}}\right]\right\rangle \ , \tag{1}$$

where $\widetilde{\mathbb{G}}$ is an extension of $\mathbb{G}$ to the cone over Σ_S – a supermanifold B_S with boundary Σ_S. The obstruction to this extension cancels due to the vanishing of $\pi_2(G)$, which still holds for a nonreductive Lie group, since any Lie group has the homotopy type of its maximal compact subgroup, which is reductive. The integrals in the above expression denote both the geometric integral over Σ and the Berezin integral over the fermionic coordinates. The subscript Ω keeps track of the dependence of the action on the metric $\Omega_{ab} = \langle X_a \, , X_b\rangle$, relative to a basis $\{X_a\}$ for $\mathfrak{g}$, which we fix once and for all. For a simple Lie group, all metrics are proportional, so one usually fixes a metric and represents the dependence on the metric by a parameter multiplying the action. Upon quantisation, this parameter usually becomes the level of the SWZW model. On the other hand, a general self-dual Lie group will have more than one metric, hence the need to keep track of it. As in the WZW model, we will assume that the metric is nondegenerate, for otherwise not all fields would be dynamical; that is, the theory would be constrained. This does not represent any loss of generality, since eliminating the non-dynamical fields, which take values in an invariant subgroup, yields a SWZW model on the quotient group which does inherit a nondegenerate metric.

The relative coefficient in (1) has been chosen for reasons which are standard [28] but which we will review below. The quantum field theory will be described by the path integral

$$Z = \int [d\mathbb{G}] e^{-I_\Omega[\mathbb{G}]} \ .$$

Independence of the quantum theory on the extension $\widetilde{\mathbb{G}}$ will in general choose

a discrete set of possible metrics (in the case of a simple Lie group, this statement is simply the quantisation of the level); although for some nonsemisimple Lie groups (e.g., if G is solvable), this will not be the case.

Although it may not be obvious at this point, the action $I_\Omega[\mathbb{G}]$ is invariant under any automorphism of the self-dual Lie group; that is, any transformation which preserves both the Lie brackets *and* the metric. For G a simple Lie group, this group is essentially $G \times G$, corresponding to left and right multiplications; but for a general self-dual Lie group, the full automorphism group may be bigger. Nevertheless we will focus in this paper only on $G \times G$. The proof that the action $I_\Omega[\mathbb{G}]$ in invariant under $G \times G$ will be delayed until Section 5 when we discuss the component action.

In fact, thanks to the choice of relative coefficients in the action, $I_\Omega[\mathbb{G}]$ enjoys an infinite-dimensional "semi-local" symmetry: $G(Z) \times G(\bar{Z})$:

$$\mathbb{G}(Z,\bar{Z}) \mapsto \Omega^{-1}(Z)\mathbb{G}(Z,\bar{Z})\bar{\Omega}(\bar{Z}) \ , \tag{2}$$

where Ω (resp. $\bar{\Omega}$) is a chiral (resp. antichiral) superfield: $\bar{D}\Omega = 0$ and $D\bar{\Omega} = 0$. This condition will in general reduce the number of components of the superfield, as well as forcing the remaining component fields to depend (anti)holomorphically on the bosonic coordinates $(z, \bar{z})$ (see Section 5).

This invariance gives rise to the following conserved currents

$$\mathbb{J}(Z) = -D\mathbb{G}\mathbb{G}^{-1}, \qquad \bar{\mathbb{J}}(\bar{Z}) = \mathbb{G}^{-1}\bar{D}\mathbb{G} \ , \tag{3}$$

satisfying the conditions $\bar{D}\mathbb{J} = D\bar{\mathbb{J}} = 0$. If we take $\mathbb{J}$ and $\bar{\mathbb{J}}$ as the dynamical variables of the SWZW model and treat the conservation laws as the equations of motion, then we obtain for the fundamental Poisson brackets

$$\{\mathbb{J}_a(Z), \mathbb{J}_b(W)\} = \left(\Omega_{ab} D_W + f_{ab}{}^c \mathbb{J}_c(W)\right) \delta(Z - W) \ ,$$

where $Z = (z, \theta)$, $W = (w, \varphi)$ and $\delta(Z - W) = \delta(z - w)(\theta - \varphi)$. Upon quantisation, the above Poisson brackets yield the current algebra encoded in the following supersymmetric operator product expansion:

$$\mathbb{J}_a(Z)\mathbb{J}_b(W) = \frac{\Omega_{ab}}{Z - W} + \frac{f_{ab}{}^c \mathbb{J}_c(W)}{(Z - W)^{1/2}} + \text{reg} \ . \tag{4}$$

(Notice that the above notation is ambiguous since the fields do not always commute with the half-integral powers of the superinterval. We will follow the convention that even though we write the fields on top of the superintervals, they are understood to appear to the right.) Similar formulas hold

for $\bar{\mathbb{J}}$. In other words, these currents satisfy an N=1 affine algebra $\widehat{\mathfrak{g}}_{N=1}$ [12], whose central extension is defined by the metric in the SWZW model. This huge symmetry underpins the exact superconformal invariance of the SWZW model. The proof of this fact relies on the N=1 Sugawara construction, to which we now turn.

2.2 The N=1 Sugawara construction

We start with the current algebra (4). By an N=1 Sugawara construction we mean the construction of an N=1 superconformal algebra out of (normal ordered) products in the supercurrents $\mathbb{J}_a(Z)$, with the property that the supercurrents are primary superfields of weight $\frac{1}{2}$. We take therefore as a general Ansatz for the energy-momentum tensor

$$\mathbb{T}(Z) = A^{ab}(D\mathbb{J}_a\mathbb{J}_b)(Z) + B^{abc}(\mathbb{J}_a(\mathbb{J}_b\mathbb{J}_c))(Z) + C^a\partial\mathbb{J}_a(Z) \ , \tag{5}$$

with A^{ab}, B^{abc}, and C^a yet unspecified coefficients. If we now impose that the $\mathbb{J}_a(Z)$ be primary superfields of weight $\frac{1}{2}$, that is,

$$\mathbb{T}(Z)\mathbb{J}_a(W) = \frac{\frac{1}{2}\mathbb{J}_a(W)}{(Z-W)^{3/2}} + \frac{\frac{1}{2}D\mathbb{J}_a(W)}{Z-W} + \frac{\partial\mathbb{J}_a(W)}{(Z-W)^{1/2}} + \text{reg} \ ,$$

we obtain that $C^a = 0$, that Ω_{ab} must be invertible with inverse Ω^{ab} and that the remaining coefficients are given by

$$A^{ab} = \tfrac{1}{2}\Omega^{ab}, \qquad B^{abc} = \tfrac{1}{6}f^{abc} \ ,$$

where we use Ω^{ab} and Ω_{ab} to raise and lower indices. A straightforward calculation then shows that the supersymmetric energy-momentum tensor (5) obeys the N=1 superconformal algebra with central charge

$$c_{\mathfrak{g}} \equiv \tfrac{3}{2}\dim\mathfrak{g} - \tfrac{1}{2}\Omega^{ab}\kappa_{ab} \ ,$$

where κ_{ab} stands for the Killing form of $\mathfrak{g}$, which for a general self-dual Lie algebra need not be nondegenerate.

Summarising, we have seen that for the N=1 Sugawara construction to exist it is necessary and sufficient that Ω_{ab}, given by (4), be invertible, which coincides with the condition required for the supersymmetric WZW model to define an unconstrained field theory. Notice that this is unlike the bosonic case, where we had two different conditions that could only be simultaneously satisfied

because of the particular structure of self-dual Lie algebras (see Theorem I.3.6 and also [7]).

3 The gauged SWZW model

After having seen that the SWZW model provides a lagrangian realisation of the N=1 Sugawara construction, we now turn our attention to the gauged SWZW model and to the superconformal field theories (SCFT) that this procedure gives rise to. We will consider here the case of the diagonal gauging for which, by manipulating the functional integral, we will be able to exhibit the resulting quantum field theory as a SCFT whose energy-momentum tensor agrees with that of an N=1 coset construction.

3.1 Gauging the SWZW model

We consider the problem of gauging a diagonal subgroup $H \subset G \times G$. In other words we want to partially promote the semi-local symmetry (2) restricted to a diagonal subgroup, to a local invariance under transformations of the form:

$$\mathbb{G}(Z, \bar{Z}) \mapsto \Lambda(Z, \bar{Z})^{-1}\, \mathbb{G}(Z, \bar{Z})\, \Lambda(Z, \bar{Z}) \ , \tag{6}$$

in the obvious notation. For this we have to introduce gauge superfields $\mathbb{A}$ and $\bar{\mathbb{A}}$, fermionic with weight $\frac{1}{2}$, with values in the complexified Lie algebra $\mathfrak{h}^{\mathbb{C}}$ of H, and which transform under gauge transformations according to

$$\begin{aligned} &\mathbb{A} \mapsto \Lambda^{-1}(D + \mathbb{A})\Lambda \ , \\ &\bar{\mathbb{A}} \mapsto \Lambda^{-1}(\bar{D} + \bar{\mathbb{A}})\Lambda \ . \end{aligned} \tag{7}$$

Gauging the SWZW model means constructing an extension $I_\Omega[\mathbb{G}, \mathbb{A}, \bar{\mathbb{A}}]$ of (1) which is invariant under (6) and (7). Using the Noether procedure we obtain

$$I_\Omega[\mathbb{G}, \mathbb{A}, \bar{\mathbb{A}}] = I_\Omega[\mathbb{G}] - 2 \int_{\Sigma_S} \left\langle \mathbb{A} \, , \bar{\mathbb{J}} \right\rangle + \left\langle \mathbb{J} \, , \bar{\mathbb{A}} \right\rangle - \left\langle \mathbb{A} \, , \bar{\mathbb{A}} \right\rangle + \left\langle \mathbb{A} \, , \mathbb{G}^{-1} \bar{\mathbb{A}} \mathbb{G} \right\rangle . \tag{8}$$

Notice that, since the gauge superfields have no kinetic term, they can be thought of as Lagrange multipliers: they introduce constraints at the level of the classical theory, which consist in setting the H-current equal to zero.

The quantum theory is described by the path integral

$$Z = \int [d\mathbb{G}][d\mathbb{A}][d\bar{\mathbb{A}}] e^{-I_\Omega[\mathbb{G},\mathbb{A},\bar{\mathbb{A}}]} \ .$$

As discussed in Section I.4 for the nonsupersymmetric case, we choose the holomorphic gauge $\bar{\mathbb{A}} = 0$; in the absence of gauge anomalies this will introduce in the gauge-fixed path integral a Faddeev-Popov determinant, $\det \bar{D}$, which can be formally expressed as the path integral

$$\det \bar{D} = \int [d\mathbb{B}][d\mathbb{C}] e^{-\int_{\Sigma_s} \langle \mathbb{B}, \bar{D}\mathbb{C}\rangle} \ ,$$

where $(\mathbb{B}, \mathbb{C})$ are $N{=}1$ Faddeev-Popov ghost superfields. $\mathbb{C}$ is a fermionic weight zero $\mathfrak{h}^{\mathbb{C}}$-valued superfield, whereas $\mathbb{B}$ is bosonic, has weight $\frac{1}{2}$ and takes its values in the dual $(\mathfrak{h}^{\mathbb{C}})^*$. The $\langle - \, , - \rangle$ above indicates the dual pairing between $\mathfrak{h}^{\mathbb{C}}$ and $(\mathfrak{h}^{\mathbb{C}})^*$, together with superfield multiplication.

The remaining gauge superfield $\mathbb{A}$ can be parametrised by $\mathbb{A} = -D\mathbb{H}\mathbb{H}^{-1}$, where $\mathbb{H}$ is an H-valued superfield. We now use the supersymmetric version of the Polyakov-Wiegmann identity [22], which holds for any self-dual Lie algebra, to express $I_\Omega[\mathbb{G}, \mathbb{A}, 0]$ in terms of the original SWZW action as follows:

$$I_\Omega[\mathbb{G}, \mathbb{A}, 0] = I_\Omega[\mathbb{G}\mathbb{H}] - I_\Omega[\mathbb{H}] \ .$$

At the quantum level, the change of variables from $\mathbb{A}$ to $\mathbb{H}$ modifies the functional measure of the path integral by a jacobian factor, $\det D_{\mathbb{A}}$, where $D_{\mathbb{A}}$ denotes the holomorphic component of the covariant derivative, $D_{\mathbb{A}} = D + [\mathbb{A}, -]$, acting on $\mathfrak{h}^{\mathbb{C}}$-valued superfields. We represent this determinant as:

$$\det D_{\mathbb{A}} = \int [d\bar{\mathbb{B}}][d\bar{\mathbb{C}}] e^{-\int_{\Sigma_s} \langle \bar{\mathbb{B}}, D_{\mathbb{A}}\bar{\mathbb{C}}\rangle}$$

where now $(\bar{\mathbb{B}}, \bar{\mathbb{C}})$ are $(\mathfrak{h}^{\mathbb{C}}, (\mathfrak{h}^{\mathbb{C}})^*)$-valued superfields. After these manipulations the path integral becomes

$$Z = \int [d\mathbb{G}][d\mathbb{H}] \, (\det D_{\mathbb{A}}) \, (\det \bar{D}) e^{-I_\Omega[\mathbb{G}\mathbb{H}] + I_\Omega[\mathbb{H}]} \ , \tag{9}$$

where in the above expression for $D_{\mathbb{A}}$ it is understood that $\mathbb{A} = -D\mathbb{H}\,\mathbb{H}^{-1}$.

We will now compute the above "determinants." We will do something a little bit more general and compute

$$\det D_{\mathbb{A}} \ \det \bar{D}_{\bar{\mathbb{A}}} \equiv \int [d\mathbb{B}][d\mathbb{C}][d\bar{\mathbb{B}}][d\bar{\mathbb{C}}] e^{-\int_{\Sigma_s} \langle \bar{\mathbb{B}}, D_{\mathbb{A}}\bar{\mathbb{C}}\rangle - \int_{\Sigma_s} \langle \mathbb{B}, \bar{D}_{\bar{\mathbb{A}}}\mathbb{C}\rangle} \ . \tag{10}$$

The above path-integral is determined by the effective action $W[\mathbb{A},\bar{\mathbb{A}}]$ defined by

$$\det D_{\mathbb{A}} \ \det \bar{D}_{\bar{\mathbb{A}}} = e^{-W[\mathbb{A},\bar{\mathbb{A}}]} \det D \ \det \bar{D} \ . \tag{11}$$

There are many ways to compute the effective action. We choose to compute it using point-splitting regularisation, or equivalently, operator product expansions. The gauge fields $\mathbb{A}$ and $\bar{\mathbb{A}}$ appear linearly in the action in the RHS of (10) coupled to the currents $\bar{\mathbb{J}}^{\mathrm{gh}}$ and $\mathbb{J}^{\mathrm{gh}}$ respectively. The integrated anomaly $W[\mathbb{A},\bar{\mathbb{A}}]$ has a nonlocal series expansion in terms of $\mathbb{A}$ and $\bar{\mathbb{A}}$ obtained by expanding the terms in the action which contain $\mathbb{A}$ and $\bar{\mathbb{A}}$. This is a nonlocal series expansion whose coefficients are correlation functions of currents. These correlation functions can be computed using the operator product expansion, provided that we understand the currents—which are composite operators—as normal ordered products regularised using point-splitting, as is usual in two-dimensional conformal field theory. This regularisation has the property that the vacuum expectation value (that is, the one-point function) of any current $\mathbb{J}^{\mathrm{gh}}$ or $\bar{\mathbb{J}}^{\mathrm{gh}}$ vanishes. Therefore the only contribution to the current-current correlators comes from the superconformal family of the identity. However it is easy to see that the superconformal family of the identity does not appear (i.e., appears with coefficient zero) in the operator product expansion of two currents. We will see this again below, but for now we simply state that the fermionic and bosonic fields in $\mathbb{B}$ and $\mathbb{C}$ contribute equally but with with opposite signs to the second order pole, which thus cancels. In summary, $W[\mathbb{A},\bar{\mathbb{A}}]$ does not depend on $(\mathbb{A},\bar{\mathbb{A}})$, and in fact,

$$\det D_{\mathbb{A}} \ \det \bar{D}_{\bar{\mathbb{A}}} = \det D \ \det \bar{D} \ .$$

Inserting this result into (9) and changing variables $\mathbb{G} \mapsto \mathbb{G}\mathbb{H}^{-1}$, which has trivial jacobian due to the absence of gauge anomalies, we arrive at the following expression:

$$Z = \int [d\mathbb{G}][d\mathbb{H}][d\mathbb{B}][d\mathbb{C}][d\bar{\mathbb{B}}][d\bar{\mathbb{C}}] e^{-I_\Omega[\mathbb{G}]+I_\Omega[\mathbb{H}]} e^{-\int \left\langle \mathbb{B},\bar{D}\mathbb{C}\right\rangle + \left\langle \bar{\mathbb{B}},D\bar{\mathbb{C}}\right\rangle} \ . \tag{12}$$

Let us pause for a moment to contemplate our result. Comparing with equation (I.4.19), we notice that the only appreciable difference is the fact that both SWZW sectors have actions corresponding to the same metric (up to signs). Furthermore similar arguments to those in Part I, show that although the three sectors appear to be independent, there exist constraints that couple them. Basically one can gauge the vector subgroup H once again in all three lagrangians, introducing *external* gauge superfields and then notice that the partition function is actually independent of the gauge fields introduced, which leads to the constraint that the supercurrent which couples to this gauge

superfield has to vanish. The total supercurrent has contributions coming from all three SCFTs.

Let us consider the holomorphic sector. The total current is given by

$$\mathbb{J}_i^{\rm tot}(Z) = \mathbb{J}_i(Z) + \widetilde{\mathbb{J}}_i(Z) + \mathbb{J}_i^{\rm gh}(Z) ~,$$

where $\{\mathbb{J}_i(Z)\}$ are the subset of the $\mathfrak{g}$ currents (3) corresponding to the subalgebra $\mathfrak{h}$. The current corresponding to the gauged sector is given by

$$\widetilde{\mathbb{J}}(Z) = -D\mathbb{H}\mathbb{H}^{-1} ~,$$

whereas the current corresponding to the ghost sector is defined by

$$\mathbb{J}_i^{\rm gh}(Z) = f_{ij}{}^k (\mathbb{B}_k \mathbb{C}^j)(Z) ~,$$

with the standard point-splitting convention for the normal ordering. These currents will satisfy three commuting current algebras with the relevant OPEs given by

$$\begin{aligned}
\mathbb{J}_i(Z)\mathbb{J}_j(W) &= \frac{\Omega_{ij}}{Z-W} + \frac{f_{ij}{}^k \mathbb{J}_k(W)}{(Z-W)^{1/2}} + \text{reg} ~, \\
\widetilde{\mathbb{J}}_i(Z)\widetilde{\mathbb{J}}_j(W) &= \frac{-\Omega_{ij}}{Z-W} + \frac{f_{ij}{}^k \widetilde{\mathbb{J}}_k(W)}{(Z-W)^{1/2}} + \text{reg} ~, \\
\mathbb{J}_i^{\rm gh}(z)\mathbb{J}_j^{\rm gh}(w) &= \frac{f_{ij}{}^k \mathbb{J}_k^{\rm gh}(W)}{(Z-W)^{1/2}} + \text{reg} ~.
\end{aligned} \tag{13}$$

Notice that as advertised above, the second order pole of the ghost currents vanishes, showing that the effective action (11) is indeed independent of the gauge fields. Adding the central extensions of the three components of the total conserved current we see that they cancel each other, which just reiterates the fact that we have gauged an anomaly-free subgroup. This guarantees that the charge which generates the BRST transformations leaving the “quantum” action invariant will square to zero. The theory resulting from the gauged SWZW model is then defined as the cohomology of the BRST operator: the states will be the BRST cohomology on states, and the fields will be the BRST cohomology on fields. In the next section we will show that among the BRST invariant fields one finds the generator of $N{=}1$ superconformal transformations on all the other BRST invariant fields—and corresponds, in fact, to the one for the SCFT based on the coset G/H. This result was first obtained independently in [23] and in [8]. As we will see in Section 7 (see

also [8]), if in addition the coset $N=1$ superconformal algebra admits an $N=2$ extension, the $N=2$ generators will also be BRST invariant.

As explained in Part I, the anti-holomorphic sector can be treated analogously by choosing the anti-holomorphic gauge $\mathbb{A} = 0$.

4 Coset SCFTs from gauged SWZW models

In this section we will analyse the quantum field theory which results after the diagonal gauging of a SWZW model based on a Lie group with a bi-invariant metric. We will prove that the resulting theory is a SCFT and that it can be identified with an $N=1$ coset construction. Indeed, we start by analysing the supersymmetric coset construction in the case of self-dual Lie algebras. For the sake of simplicity we will only consider the holomorphic sector, the treatment of the antiholomorphic sector being completely analogous.

4.1 The $N=1$ coset construction

We start with the $N=1$ affine algebra $\widehat{\mathfrak{g}}_{N=1}$ (4) based on the self-dual Lie algebra $(\mathfrak{g}, \Omega)$. We consider a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and we fix a basis for it $\{X_i\}$ which we can think of as a sub-basis of the chosen basis $\{X_a\}$ for $\mathfrak{g}$. Then the $N=1$ affine algebra $\widehat{\mathfrak{h}}$ of the $\{\mathbb{J}_i\}$ currents is given by

$$\mathbb{J}_i(Z)\mathbb{J}_j(W) = \frac{\Omega_{ij}}{Z-W} + \frac{f_{ij}{}^k \mathbb{J}_k(W)}{(Z-W)^{1/2}} + \text{reg} \ ,$$

where Ω_{ij} are the entries of the restriction $\Omega|_{\mathfrak{h}}$ of Ω to $\mathfrak{h} \subset \mathfrak{g}$.

Clearly, a coset construction only exists if $\mathfrak{h}$ itself admits a supersymmetric Sugawara construction based on the above currents. This means, in light of Section 2, that the bilinear form $\Omega|_{\mathfrak{h}}$ has to be nondegenerate. In this case we can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ which, because of the invariance of the metric, is not just a decomposition of vector spaces but also one of $\mathfrak{h}$-modules. In other words, if we now let $\{X_\alpha\}$ denote a basis for $\mathfrak{h}^{\perp}$, we can summarise this discussion by saying that $\Omega_{i\alpha} = 0$ and that $f_{i\alpha}{}^j = 0$. This fact will play an important role repeatedly (albeit tacitly at times) in the rest of this paper.

With these remarks behind us, we see that the supersymmetric energy-momentum tensor corresponding to $\mathfrak{h}$ reads

$$\mathbb{T}_{\mathfrak{h}}(Z) = \tfrac{1}{2}\Omega^{ij}(D\mathbb{J}_i\mathbb{J}_j)(Z) + \tfrac{1}{6}f^{ijk}(\mathbb{J}_i(\mathbb{J}_j\mathbb{J}_k))(Z) \ ,$$

and generates an N=1 superconformal algebra with central charge

$$c_{\mathfrak{h}} = \tfrac{3}{2}\dim\mathfrak{h} - \tfrac{1}{2}\Omega_{\mathfrak{h}}^{ij}\kappa_{ij}^{\mathfrak{h}} \ .$$

A straightforward calculation then shows that the coset energy-momentum tensor defined by

$$\mathbb{T}_{\mathfrak{g}/\mathfrak{h}} \equiv \mathbb{T}_{\mathfrak{g}} - \mathbb{T}_{\mathfrak{h}} \tag{14}$$

commutes with $\mathbb{T}_{\mathfrak{h}}$ and generates an N=1 superconformal theory with central charge

$$c_{\mathfrak{g}/\mathfrak{h}} \equiv c_{\mathfrak{g}} - c_{\mathfrak{h}} = \tfrac{3}{2}\left(\dim\mathfrak{g} - \dim\mathfrak{h}\right) - \tfrac{1}{2}\left(\kappa_{ab}^{\mathfrak{g}}\Omega^{ab} - \kappa_{ij}^{\mathfrak{h}}\Omega^{ij}\right) \ . \tag{15}$$

As explained in Part I, it follows from the structure theorem of self-dual Lie algebras (see [7] and references therein) that, just as in the reductive case, there exist natural cosets associated to nonreductive self-dual Lie algebras. We hope to return to the explicit construction of these cosets in a future publication.

4.2 The SCFTs in the gauged SWZW model

We will now show that the gauged SWZW model described in the previous section is a superconformal field theory whose energy momentum tensor agrees with the coset energy-momentum tensor. We saw that the quantum field theory of the gauged WZW model is given by three quantum field theories coupled by a constraint which we can analyse in the BRST formalism. As we now show, each of the three sectors of the theory is superconformal. We will also see that the total energy-momentum tensor is BRST invariant and indeed BRST-equivalent to the N=1 coset energy momentum tensor in (14).

We start then with the SWZW SCFT with group G and metric Ω_{ab}. This component corresponds to the original (ungauged) SWZW model which we discussed in the first part of Section 2. There we have seen that we have a set of supercurrents $\{\mathbb{J}_a(Z)\}_{a=1}^{\dim\mathfrak{g}}$ whose OPE is given by (4). We have also seen that according to the N=1 self-dual Sugawara construction this SWZW sector does give rise to a SCFT if and only if this metric is nondegenerate. In this case, the energy-momentum tensor

$$\mathbb{T}_{\mathfrak{g}}(Z) = \tfrac{1}{2}\Omega^{ab}(D\mathbb{J}_a\mathbb{J}_b)(Z) + \tfrac{1}{6}f^{abc}(\mathbb{J}_a(\mathbb{J}_b\mathbb{J}_c))(Z) \ , \tag{16}$$

obeys a Virasoro algebra with the central charge

$$c_{\mathfrak{g}} = \tfrac{3}{2}\dim\mathfrak{g} - \tfrac{1}{2}\Omega^{ab}\kappa^{\mathfrak{g}}_{ab} \ . \tag{17}$$

The next ingredient is provided by the SWZW model with group $H \subset G$ and metric $-\Omega_{ij}$. This is characterised by the set of currents $\{\widetilde{\mathbb{J}}_i(Z)\}_{i=1}^{\dim\mathfrak{h}}$ whose OPE is given by (13). Applying (once again) the argument of Section 2, we get that this current algebra gives rise to a SCFT if and only if the restriction of Ω to $\mathfrak{h}$ is itself nondegenerate. In this case the corresponding energy-momentum tensor

$$\widetilde{\mathbb{T}}_{\mathfrak{h}}(Z) = -\tfrac{1}{2}\Omega^{ij}(D\widetilde{\mathbb{J}}_i\widetilde{\mathbb{J}}_j)(Z) + \tfrac{1}{6}f^{ijk}(\widetilde{\mathbb{J}}_i(\widetilde{\mathbb{J}}_j\widetilde{\mathbb{J}}_k))(Z) \ , \tag{18}$$

will generate an $N{=}1$ superconformal algebra with central charge

$$\widetilde{c}_{\mathfrak{h}} = \tfrac{3}{2}\dim\mathfrak{h} + \tfrac{1}{2}\Omega^{ij}\kappa^{\mathfrak{h}}_{ij} \ . \tag{19}$$

The last sector of the theory consists of a set of $(\dim\mathfrak{h})$ supersymmetric $(\mathbb{B},\mathbb{C})$ systems of superconformal weights $(\frac{1}{2},0)$ respectively, with OPE given by

$$\mathbb{B}_i(Z)\mathbb{C}^j(W) = \frac{\delta_i{}^j}{(Z-W)^{1/2}} + \text{reg} \ .$$

The energy-momentum tensor for this $(\mathbb{B},\mathbb{C})$ system has the standard form

$$\mathbb{T}_{\text{gh}}(Z) = -\tfrac{1}{2}\left(\mathbb{B}_i\partial\mathbb{C}^i\right)(Z) + \tfrac{1}{2}\left(D\mathbb{B}_iD\mathbb{C}^i\right)(Z) \ , \tag{20}$$

and obeys the superconformal algebra with central charge

$$c_{\text{gh}} = -3\dim\mathfrak{h} \ . \tag{21}$$

Finally, we introduce the last ingredient of this theory: the BRST current:

$$j(Z) = (\mathbb{C}^i(\mathbb{J}_i + \widetilde{\mathbb{J}}_i + \tfrac{1}{2}\mathbb{J}^{\text{gh}}_i))(Z) \ . \tag{22}$$

It follows from the formulae in Appendix A, that the BRST variation $d\mathbb{A}(Z)$ of a superfield $\mathbb{A}(Z)$ is given by:

$$d\mathbb{A} \equiv [\![j,\mathbb{A}]\!]_{\frac{1}{2}} \ .$$

It follows using the associativity axiom that

$$d^2\mathbb{A} = \tfrac{1}{2}[\![[\![j,j]\!]_{\frac{1}{2}},\mathbb{A}]\!]_{\frac{1}{2}} \ ;$$

whence, in the absence of any further relations, $d^2 = 0$ provided that $[\![j, j]\!]_{\frac{1}{2}} = D\mathbb{X}$ for some $\mathbb{X}$. Indeed it is an easy computation using the formulae in Appendix A, that actually $[\![j, j]\!]_{\frac{1}{2}} = 0$, which simply reiterates the fact that we have gauged an anomaly-free subgroup and justifies our assuming the absence of gauge anomalies.

4.3 The energy-momentum tensor

The total energy-momentum tensor is given by the sum of the three commuting terms given by (16), (18), and (20):

$$\mathbb{T}(z) = \mathbb{T}_{\mathfrak{g}}(z) + \widetilde{\mathbb{T}}_{\mathfrak{h}}(z) + \mathbb{T}_{\rm gh}(z) ,$$

whose central charge is obtained by adding up (17), (19), and (21):

$$c = \left(\tfrac{3}{2}\dim\mathfrak{g} - \tfrac{1}{2}\Omega^{ab}\kappa^{\mathfrak{g}}_{ab}\right) - \left(\tfrac{3}{2}\dim\mathfrak{h} - \tfrac{1}{2}\Omega^{ij}\kappa^{\mathfrak{h}}_{ij}\right) .$$

Notice that this agrees with the coset central charge given by equation (15). This prompts us to compare $\mathbb{T}(z)$ with the energy-momentum tensor of the corresponding coset construction. We introduce for this purpose

$$\mathbb{T}_{\mathfrak{h}} = \tfrac{1}{2}\Omega^{ij}(D\mathbb{J}_i\mathbb{J}_j) + \tfrac{1}{6}f^{ijk}(\mathbb{J}_i(\mathbb{J}_j\mathbb{J}_k)) .$$

Our total energy-momentum tensor then splits into a sum of two *commuting* terms

$$\mathbb{T}(z) = \mathbb{T}_{\mathfrak{g}/\mathfrak{h}}(z) + \mathbb{T}'(z) ,$$

with $\mathbb{T}_{\mathfrak{g}/\mathfrak{h}}(z)$ being the coset energy-momentum tensor defined by (14) and

$$\mathbb{T}'(z) = \mathbb{T}_{\mathfrak{h}}(z) + \widetilde{\mathbb{T}}_{\mathfrak{h}}(z) + \mathbb{T}_{\rm gh}(z) .$$

Moreover, a short computation shows us that $\mathbb{T}'$ satisfies an $N{=}1$ superconformal algebra with vanishing central charge, $c' = 0$.

Our aim now is to show that $\mathbb{T}$, $\mathbb{T}_{\mathfrak{g}/\mathfrak{h}}$ and $\mathbb{T}'$ are BRST-invariant, so that they are physical operators (that is, they induce operators in the physical space). Moreover we will show that $\mathbb{T}'$ is BRST-trivial, which means that it acts trivially on physical states.

To this effect it is convenient to list the following identities:

$$
\begin{aligned}
d\mathbb{B}_i(Z) &= \mathbb{J}_i^{\text{tot}}(Z)\ , \\
d\mathbb{C}^i(Z) &= \tfrac{1}{2} f_{jk}{}^i \mathbb{C}^j \mathbb{C}^k(Z)\ , \\
d\mathbb{J}_a(Z) &= \Omega_{aj} D\mathbb{C}^j(Z) + f_{aj}{}^b \mathbb{C}^j \mathbb{J}_b(Z)\ , \\
d\widetilde{\mathbb{J}}_i(Z) &= -\Omega_{ij} D\mathbb{C}^j(Z) + f_{ij}{}^k \mathbb{C}^j \widetilde{\mathbb{J}}_k(Z)\ .
\end{aligned}
$$

Using these relations we deduce the following

$$
d\mathbb{J}_i^{\text{gh}}(Z) = f_{ij}{}^k (\mathbb{J}_k + \widetilde{\mathbb{J}}_k)\mathbb{C}^j - f_{ik}{}^m f_{j\ell}{}^k \mathbb{B}_m (\mathbb{C}^j \mathbb{C}^\ell)\ .
$$

We can now use these identities and the fact that d is an odd derivation over the normal ordered product, to prove that:

$$
d\mathbb{T}(Z) = d\mathbb{T}_{\mathfrak{g}/\mathfrak{h}}(Z) = d\mathbb{T}'(Z) = 0\ ;
$$

that is, $\mathbb{T}(Z)$, $\mathbb{T}_{\mathfrak{g}/\mathfrak{h}}(Z)$ and $\mathbb{T}'(Z)$ are BRST-invariant. Furthermore, it also follows that there exists an operator

$$
\begin{aligned}
\mathbb{O}(Z) = &-\tfrac{1}{4}\Omega^{ij} D\mathbb{B}_i (\mathbb{J}_j - \widetilde{\mathbb{J}}_j) + \tfrac{1}{4}\Omega^{ij}\mathbb{B}_i (D\mathbb{J}_j - D\widetilde{\mathbb{J}}_j) \\
&+ \tfrac{1}{6} f^{ijk} \mathbb{B}_i (\mathbb{J}_j \mathbb{J}_k + \widetilde{\mathbb{J}}_j \widetilde{\mathbb{J}}_k - \mathbb{J}_j \widetilde{\mathbb{J}}_k)
\end{aligned}
$$

such that

$$
\mathbb{T}'(Z) = d\mathbb{O}(Z)\ ;
$$

in other words, $\mathbb{T}'(Z)$ is BRST trivial, whence it is zero in cohomology.

These results imply that the quantum field theory defined by the gauged SWZW model (12) is superconformal, with energy-momentum tensor given by $\mathbb{T}(Z)$ and its antiholomorphic counterpart $\bar{\mathbb{T}}(\bar{Z})$. Moreover $\mathbb{T}(Z)$ (and similarly for $\bar{\mathbb{T}}(\bar{Z})$) is precisely the coset energy-momentum tensor $\mathbb{T}_{\mathfrak{g}/\mathfrak{h}}(Z)$; whence we conclude that the gauged SWZW model defines a superconformal field theory which can be identified with the supersymmetric coset G/H.

5 The SWZW model in components

In the last three sections we have seen that it is possible to consistently define the $N{=}1$ supersymmetric WZW model associated to a general self-dual Lie group (both gauged and "as is") in superspace, that it yields a superconformal invariant theory, and moreover we have exactly determined the SCFT it describes. While this is certainly a satisfactory state of affairs, we are motivated

to consider the formulation of the SWZW model in components because of two main reasons. On the one hand, there has been quite a lot of work done in components and we think it is useful to clarify the correspondence between the two approaches; especially when it comes to the gauged SWZW model. And on the other hand, in the discussion on N=2 Kazama–Suzuki cosets in Section 7, we will be forced to work in components, since there is no local expression of the extra supersymmetry generators in terms of superfields.

We start this section by exhibiting the underlying Lie supergroup structure in the SWZW model. To be precise the superfield $\mathbb{G}(Z,\bar{Z})$ does not take values in the Lie group G but rather in a Lie supergroup with body G. This Lie supergroup possesses an invariant metric and the symmetries of the SWZW model arise as symmetries of this structure in a manifest manner. Moreover the underlying supergroup will also suggest a natural parametrisation of the superfield $\mathbb{G}(Z,\bar{Z})$ into components.

5.1 The underlying Lie supergroup

The superfield $\mathbb{G}(Z,\bar{Z})$ can be parametrised in several ways. A parametrisation which teaches us where $\mathbb{G}$ really takes values, is the following. Let $\mathbb{X}(Z,\bar{Z})$ be a superfield given by

$$\mathbb{X}(Z,\bar{Z}) = x(z,\bar{z}) + \theta\chi(z,\bar{z}) + \bar{\theta}\bar{\chi}(z,\bar{z}) + \theta\bar{\theta}f(z,\bar{z}) \ ,$$

where x, f are bosonic and $\chi, \bar{\chi}$ are fermionic fields with values in the Lie algebra $\mathfrak{g}$ of G. It would be tempting to call $\mathbb{X}$ a $\mathfrak{g}$-valued superfield, but because of the θ's this is strictly speaking not correct. Instead, $\mathbb{X}$ takes values in a Lie superalgebra $\mathfrak{s}$ built out of $\mathfrak{g}$ in the following canonical fashion. As a vector space $\mathfrak{s}$ consists of four copies of $\mathfrak{g}$. If we write $\mathfrak{s}_{\bar{0}}$ and $\mathfrak{s}_{\bar{1}}$ for the even and odd subspaces of $\mathfrak{s}$, then as vector spaces both are isomorphic to $\mathfrak{g}\oplus\mathfrak{g}$. If $\mathbb{X}$ is as above then $(x,f)\in\mathfrak{s}_{\bar{0}}$ and $(\chi,\bar{\chi})\in\mathfrak{s}_{\bar{1}}$. The Lie bracket of $\mathfrak{s}$ is induced from the one in $\mathfrak{g}$. Indeed, if $\mathbb{X}$ is as above and $\mathbb{Y} = y + \theta\gamma + \bar{\theta}\bar{\gamma} + \theta\bar{\theta}g$, then

$$\begin{aligned}[\mathbb{X},\mathbb{Y}] = {}& [x,y] + \theta\left([x,\gamma] - [y,\chi]\right) + \bar{\theta}\left([x,\bar{\gamma}] - [y,\bar{\chi}]\right)\\ &+ \theta\bar{\theta}\left([x,g] - [y,f] - [\chi,\bar{\gamma}] + [\bar{\chi},\gamma]\right) \ ,\end{aligned}$$

whence we see that, for instance, $\mathfrak{g}\subset\mathfrak{s}_{\bar{0}}$ is naturally a subalgebra, and that in fact $\mathfrak{s}_{\bar{0}}$ is the abelian extension $\mathfrak{g}\ltimes\mathfrak{g}_{\mathrm{ab}}$, where $\mathfrak{g}$ acts on $\mathfrak{g}_{\mathrm{ab}}$ via the adjoint representation, and $\mathfrak{g}_{\mathrm{ab}}$ is $\mathfrak{g}$ made abelian. Similarly $\mathfrak{g}\subset\mathfrak{s}_{\bar{0}}$ acts on $\mathfrak{s}_{\bar{1}}$ via two copies of the adjoint representation.

The metric on $\mathfrak{g}$ induces a metric on $\mathfrak{s}$ in the following way. if $\mathbb{X}$ and $\mathbb{Y}$ are as above, we write

$$\begin{aligned}\langle \mathbb{X} \,, \mathbb{Y}\rangle = \langle x \,, y\rangle &+\theta\left(\langle x \,, \gamma\rangle - \langle y \,, \chi\rangle\right) + \bar\theta\left(\langle x \,, \bar\gamma\rangle - \langle y \,, \bar\chi\rangle\right)\\ &+\theta\bar\theta\left(\langle x \,, g\rangle - \langle y \,, f\rangle - \langle \chi \,, \bar\gamma\rangle + \langle \bar\chi \,, \gamma\rangle\right)\ .\end{aligned}$$

Each component is $\mathfrak{s}$-invariant, but only the $\theta\bar\theta$-component is nondegenerate. Notice that this component is precisely what is projected out by the Berezin integral $\int_B \langle \mathbb{X} \,, \mathbb{Y}\rangle$. In other words the composition of Berezin integral and the metric on $\mathfrak{g}$ define a metric on $\mathfrak{s}$ which is clearly invariant. In other words, $\mathfrak{s}$ is self-dual.

As discussed in Part I (see also [7]) any indecomposable self-dual Lie algebra which is not simple or one-dimensional is obtained as a double extension of a self-dual Lie algebra by either a one-dimensional or a simple Lie algebra. As remarked in [7], although double extensions still produce self-dual Lie superalgebras, there exists no proof of (nor a counterexample against) the similar result for self-dual Lie superalgebras. We can check in this case that $\mathfrak{s}$ is a double of extension of the abelian Lie superalgebra $\mathfrak{s}_{\bar 1} = \mathfrak{g} \oplus \mathfrak{g}$ with off-diagonal metric:

$$\langle (\chi \,, \bar\chi), (\gamma, \bar\gamma)\rangle = \langle \bar\chi \,, \gamma\rangle - \langle \chi \,, \bar\gamma\rangle\ ,$$

by the Lie algebra $\mathfrak{g}$, acting under two copies of the adjoint representation.

If we now parametrise $\mathbb{G}(Z,\bar Z) = \exp \mathbb{X}(Z,\bar Z)$ we see that the superfield $\mathbb{G}(Z,\bar Z)$ describing the SWZW model in equation (1), takes values in a Lie supergroup S with Lie superalgebra $\mathfrak{s}$. It now becomes obvious that the SWZW action (1) possesses the symmetries ascribed to it in Section 2. Just as the symmetries of the ordinary WZW model follow from the fact that G is a self-dual Lie group, the symmetries of the SWZW model follow from the fact that S is a self-dual Lie supergroup.

Since the Lie supergroup S has G as its body, it follows from the structure theory of Lie supergroups that we can decompose elements in S as follows:

$$S = G \cdot S_N$$

where S_N are those group elements in S of the form $\mathbf{1}+$ nilpotent. In other words, elements of S_N can be understood as exponentials of elements $\mathbb{X}_N$ of $\mathfrak{s}$ of the form $\mathbb{X}_N = \theta\lambda + \bar\theta\bar\lambda + \theta\bar\theta f$. Hence this decomposition suggests a different parametrisation of the superfield $\mathbb{G}(Z,\bar Z)$:

$$\mathbb{G} = g\,\mathbb{G}_N = g\left[1 + \theta\lambda + \bar\theta\lambda + \theta\bar\theta(f - \tfrac{1}{2}\lambda\bar\lambda + \tfrac{1}{2}\bar\lambda\lambda)\right]\ , \tag{23}$$

where g takes values in the Lie group G, and λ, $\bar\lambda$ and f all take values in the Lie algebra. Notice however that because λ and $\bar\lambda$ are odd, what appears in

the $\theta\bar{\theta}$-term is not the Lie bracket $[\lambda, \bar{\lambda}]$ but the anticommutator. Hence the $\theta\bar{\theta}$-component of $\mathbb{G}_N$ does not live in the Lie algebra $\mathfrak{g}$ but in $\mathfrak{g} \oplus S^2\mathfrak{g}$.

5.2 The component action for the SWZW model

In order to write the SWZW action in components we start with (1) and parametrise the superfield $\mathbb{G}$ in a more symmetric version of (23):

$$\begin{aligned} \mathbb{G} &= g\left[1 + \theta\psi + \bar{\theta}g^{-1}\bar{\psi}g + \theta\bar{\theta}(a - \psi g^{-1}\bar{\psi}g)\right] \ , \\ \mathbb{G}^{-1} &= \left[1 - \theta\psi - \bar{\theta}g^{-1}\bar{\psi}g - \theta\bar{\theta}(a - g^{-1}\bar{\psi}g\psi)\right] g^{-1} \ , \end{aligned} \tag{24}$$

where $g : \Sigma \to G$ and all the other fields are defined on Σ and take values in the Lie algebra $\mathfrak{g}$. As usual in supersymmetric σ-models, the fermions are sections of the spinor bundle on Σ twisted by the pull-back of the tangent bundle of the Lie group TG. But here, in addition, there is an implicit trivialisation of the tangent bundle of the Lie group, so that ψ and $\bar{\psi}$ are actually Lie algebra valued. This trivialisation is accomplished via left or right translations. One choice is natural for ψ and another for $\bar{\psi}$ which explains the seemingly asymmetrical way in which they appear in (24). We will have more to say about this parametrisation below.

Introducing the expression of the superfield (24) in the SWZW action, eliminating the auxiliary field a by its equation of motion $a = 0$, and after some algebra we obtain:

$$I_\Omega[g, \psi, \bar{\psi}] = I_\Omega[g] - \int_\Sigma \left\langle \psi \, , \bar{\nabla}\psi \right\rangle + \left\langle \bar{\psi} \, , \nabla\bar{\psi} \right\rangle \ , \tag{25}$$

where $I_\Omega[g]$ is the WZW action given by (I.2.11), and the last two terms describe describe Weyl fermions in the adjoint representation of $\mathfrak{g}$, axially coupled to the bosonic field g via the covariant derivatives:

$$\begin{aligned} \nabla &\equiv \mathrm{ad}_g \circ \partial \circ \mathrm{ad}_g^{-1} = \partial - [\partial g \ g^{-1}, -] \ , \\ \bar{\nabla} &\equiv \mathrm{ad}_g^{-1} \circ \bar{\partial} \circ \mathrm{ad}_g = \bar{\partial} + [g^{-1}\bar{\partial} g, -] \ , \end{aligned}$$

with ad_g denoting the adjoint action of the group G on its Lie algebra $\mathfrak{g}$:

$$\mathrm{ad}_g(X) = g \, X \, g^{-1} \ .$$

The above action is clearly supersymmetric. Indeed, one can easily check that (25) is invariant under the following supersymmetry transformations:

$$
\begin{aligned}
\delta g &= \epsilon g\psi + \bar{\epsilon}\bar{\psi}g \ , \\
\delta\psi &= \epsilon(g^{-1}\partial g - \psi^2) - \bar{\epsilon}[\psi, g^{-1}\bar{\psi}g] \ , \\
\delta\bar{\psi} &= \epsilon[g\psi g^{-1}, \bar{\psi}] + \bar{\epsilon}(\bar{\partial}g g^{-1} + \bar{\psi}^2) \ .
\end{aligned}
$$

In addition, the SWZW action (25) is invariant under both bosonic and fermionic symmetry transformations which can be determined either by breaking (2) into components, or by direct investigation of (25). The bosonic symmetry transformations read

$$
\begin{aligned}
g(z,\bar{z}) &\mapsto h^{-1}(z)g(z,\bar{z})\bar{h}(\bar{z}) && (26)\\
\psi(z,\bar{z}) &\mapsto \bar{h}^{-1}(\bar{z})\psi(z,\bar{z})\bar{h}(\bar{z}) && (27)\\
\bar{\psi}(z,\bar{z}) &\mapsto h^{-1}(z)\bar{\psi}(z,\bar{z})h(z) \ , && (28)
\end{aligned}
$$

with h and $\bar{h}$ being holomorphic and antiholomorphic maps, respectively, from Σ to the group G. The bosonic conserved currents associated to this invariance are

$$
I(z) = -(\partial g \ g^{-1} + g\psi^2 g^{-1}) \ , \qquad \bar{I}(\bar{z}) = (g^{-1}\bar{\partial}g - g^{-1}\bar{\psi}^2 g) \ ,
$$

with the equations of motion $\bar{\partial}I = \partial\bar{I} = 0$. Notice the structure of these currents: the first term is nothing but the conserved current of the bosonic WZW model, and they have a second term due to the fermions, which depends on the structure constant of the algebra $\mathfrak{g}$. The fermionic symmetries leave g inert and act on the fermions as follows:

$$
\begin{aligned}
\psi(z,\bar{z}) &\mapsto \psi(z,\bar{z}) + g^{-1}(z,\bar{z})\chi(z)g(z,\bar{z}) && (29)\\
\bar{\psi}(z,\bar{z}) &\mapsto \bar{\psi}(z,\bar{z}) + g(z,\bar{z})\bar{\chi}(\bar{z})g^{-1}(z,\bar{z}) \ , && (30)
\end{aligned}
$$

with χ, $\bar{\chi}$ holomorphic and antiholomorphic maps, respectively, from Σ to $\mathfrak{g}$. The fermionic conserved currents corresponding to this invariance are

$$
\Psi(z) = -g\psi g^{-1} \ , \qquad \bar{\Psi}(\bar{z}) = g^{-1}\bar{\psi}g \ ,
$$

with the equations of motion $\bar{\partial}\Psi = \partial\bar{\Psi} = 0$.

One can easily verify that the above conserved currents coincide with the components of the supercurrents $\mathbb{J}$, $\bar{\mathbb{J}}$ (after imposing the equations of motion), that is

$$
\mathbb{J}(Z) = \Psi(z) + \theta I(z) \ , \qquad \bar{\mathbb{J}}(\bar{Z}) = \bar{\Psi}(\bar{z}) + \bar{\theta}\bar{I}(\bar{z}) \ .
$$

Therefore it should not come as a surprise that the current algebra generated by these currents coincides with the one obtained by breaking (4) into components. Indeed, if we take I, $\bar{I}$ and Ψ, $\bar{\Psi}$ as the dynamical variables and compute the fundamental Poisson brackets we find:

$$\begin{aligned}
\{I_a(z), I_b(w)\} &= \left(\Omega_{ab}\partial_w + f_{ab}{}^c I_c(w)\right)\delta(z-w)\\
\{I_a(z), \Psi_b(w)\} &= f_{ab}{}^c \Psi_c(w)\delta(z-w)\\
\{\Psi_a(z), \Psi_b(w)\} &= \Omega_{ab}\delta(z-w)\ ,
\end{aligned}$$

which upon quantisation become:

$$I_a(z)I_b(w) = \frac{\Omega_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c I_c(w)}{z-w} + \mathrm{reg} \tag{31}$$

$$I_a(z)\Psi_b(w) = \frac{f_{ab}{}^c \Psi_c(w)}{z-w} + \mathrm{reg} \tag{32}$$

$$\Psi_a(z)\Psi_b(w) = \frac{\Omega_{ab}}{z-w} + \mathrm{reg}\ , \tag{33}$$

and similar formulas for the antiholomorphic currents.

5.3 The quantum theory

The quantum theory is described by the path integral

$$Z = \int [dg][d\psi][d\bar{\psi}] e^{-I_\Omega[g] + \int \langle \psi, \bar{\nabla}\psi\rangle + \langle \bar{\psi}, \nabla\bar{\psi}\rangle}\ . \tag{34}$$

One can of course decouple the fermions in (34) by performing an axial gauge transformation; this will incur in a nontrivial jacobian in the path integral which is described by the effective action, $W[g] = -I_{\frac{1}{2}\kappa}[g]$, which is nothing but a WZW action on G, with a metric proportional to the Killing metric on $\mathfrak{g}$. Thus, in terms of the free fermions the path integral reads

$$Z = \int [dg][d\psi][d\bar{\psi}] e^{-I_{\Omega-\frac{1}{2}\kappa}[g] + \int \langle \psi, \bar{\partial}\psi\rangle + \langle \bar{\psi}, \partial\bar{\psi}\rangle}\ . \tag{35}$$

As shown in Part I (see also [7]) the bosonic action is still generically unconstrained since for a generic self-dual Lie algebra $\Omega - \frac{1}{2}\kappa$ is nondegenerate (see Theorem I.3.6 for the precise statement). The proof of this result given in Part I relies strongly on the structure theory of self-dual Lie algebras. In the light of the above discussion and of equation (35), a heuristic argument for

the nondegeneracy of $\Omega - \frac{1}{2}\kappa$ might be obtained simply by supersymmetrising the WZW model, and then decoupling the fermions.

At the level of the current algebra, decoupling the fermions amounts to a redefinition of the bosonic current. In the last paragraph we have written the N=1 current algebra in terms of a basis of currents $\{I_a, \Psi_a\}$. If we now redefine the bosonic current

$$J_a \equiv I_a - \tfrac{1}{2}\Omega^{bd} f_{ab}{}^c (\Psi_c \Psi_d) \ , \tag{36}$$

and leave the fermionic current unmodified, then the fermions decouple:

$$J_a(z) J_b(w) = \frac{\Omega_{ab} - \frac{1}{2}\kappa_{ab}}{(z-w)^2} + \frac{f_{ab}{}^c J_c(w)}{z-w} + \text{reg} \ , \tag{37}$$

$$J_a(z) \Psi_b(w) = \text{reg} \ , \tag{38}$$

$$\Psi_a(z) \Psi_b(w) = \frac{\Omega_{ab}}{z-w} + \text{reg} \ . \tag{39}$$

In other words the modified bosonic currents commute with the fermionic ones, while the residue of the double pole in the OPE of the bosonic current with itself receives a shift proportional to the Killing metric. Notice that the central extension in the OPE of the bosonic currents corresponds exactly to the metric of the bosonic part of the SWZW action, and similarly the central extension of the fermionic OPE corresponds to the metric of the fermionic part of the action.

This decoupled basis is particularly convenient for writing the superconformal algebra in components. Indeed if we define the components of the supersymmetric generator of the superconformal algebra by

$$\mathbb{T}(Z) = \tfrac{1}{2}\mathsf{G}(z) + \theta \mathsf{T}(z) \ ,$$

then we get

$$\begin{aligned} \mathsf{T}(z) &= \tfrac{1}{2}\Omega^{ab}(J_a J_b)(z) + \tfrac{1}{2}\Omega^{ab}(\partial\Psi_a \Psi_b)(z) \ , \\ \mathsf{G}(z) &= \Omega^{ab}(J_a \Psi_b)(z) - \tfrac{1}{6} f^{abc}(\Psi_a(\Psi_b \Psi_c))(z) \ . \end{aligned}$$

Notice that in this basis the energy-momentum tensor $\mathsf{T}(z)$ is written as a sum of two independent terms, the first one being the bosonic Sugawara energy-momentum tensor corresponding to the bosonic current algebra (37), whereas the second one is the standard energy-momentum of $\dim G$ free fermions.

Let us conclude this section with a brief discussion on the choice of parametrisation made at the beginning of this section. It is clear that the parametrisation

chosen for the superfield $\mathbb{G}$ in (24) is not unique. Indeed, one can redefine the fermionic fields in such a way that the fermions in the SWZW action end up either coupled or uncoupled to the bosonic field g. This naturally raises the question whether these different parametrisations are equivalent or not, and if they are not, how does one choose the "right" parametrisation.

At the classical level, there is no real distinction between them. Indeed, let us consider for concreteness the following two parametrisations: the one defined by (24) which gives rise to the coupled fermions in the (component) SWZW action, and the parametrisation obtained from (24) by performing the following axial gauge transformation

$$\psi \mapsto g^{-1}\psi g \qquad \text{and} \qquad \bar{\psi} \mapsto g\psi g^{-1} \ .$$

This yields a component action which is simply the sum of the bosonic action and the free fermions, and which is thus superconformally invariant.

The situation at the quantum level is slightly different, and the free and the coupled parametrisations are no longer equivalent. The reason being that the change of variables which decouple the fermions classically, gives rise to a nontrivial jacobian at the quantum level. More explicitly one can see this by simply comparing the "decoupled" path integral (35) with the path integral corresponding to the "free" parametrisation:

$$Z' = \int [dg][d\psi][d\bar{\psi}] e^{-I_\Omega[g] + \int \left\langle \psi, \bar{\partial}\psi \right\rangle + \left\langle \bar{\psi}, \partial\bar{\psi} \right\rangle} \ . \tag{40}$$

The two parametrisations give rise to two different quantum theories characterised by different metrics in the bosonic part of the action.

Which is then the correct parametrisation? Keeping in mind that we are interested in finding a lagrangian description for certain classes of SCFTs, a natural choice of parametrisation is one which yields a (manifestly) superconformal theory; in other words, a SWZW model whose symmetry algebra is an N=1 affine Lie algebra. We have already seen that, by using the coupled parametrisation (24), we obtain the right current algebra. On the other hand, if we would take the path integral (40) and we write down the corresponding current algebra we see immediately that it does not agree with the N=1 affine algebra which we need for the Sugawara construction. (In the decoupled basis, both the bosonic and the fermionic OPEs have the unshifted metric as central term.) In summary, this justifies our choice of parametrisation of the superfield $\mathbb{G}$ in (24).

6 The gauged SWZW models in components and Witten's action

We have obtained two different actions for the SWZW model: in superfields and in components. Moreover we have obtained a superfield action for the gauged SWZW model. We can obtain a component action in either of two ways: we can gauge the component action (25) or we can break the superfield gauged action (8) into components. In this section we show that both methods yield equivalent actions, and that these actions are in turn equivalent to the action written down by Witten in [29], in his formulation of the topological Kazama–Suzuki models.

6.1 Gauging the component action

We now set out to gauge *both* the fermionic (29)-(30) and bosonic (26)-(28) symmetries of the component action of the SWZW model. In other words, we will now construct an extension of the action (25) which is invariant under bosonic transformations of the form

$$\begin{aligned} g(z,\bar{z}) &\mapsto \lambda^{-1}(z,\bar{z})g(z,\bar{z})\lambda(z,\bar{z}) \\ \psi(z,\bar{z}) &\mapsto \lambda^{-1}(z,\bar{z})\psi(z,\bar{z})\lambda(z,\bar{z}) \\ \bar{\psi}(z,\bar{z}) &\mapsto \lambda^{-1}(z,\bar{z})\bar{\psi}(z,\bar{z})\lambda(z,\bar{z}) \ , \end{aligned}$$

and also under the fermionic transformations

$$\begin{aligned} \psi(z,\bar{z}) &\mapsto \psi(z,\bar{z}) + g^{-1}(z,\bar{z})\chi(z,\bar{z})g(z,\bar{z}) \\ \bar{\psi}(z,\bar{z}) &\mapsto \bar{\psi}(z,\bar{z}) + g(z,\bar{z})\bar{\chi}(z,\bar{z})g^{-1}(z,\bar{z}) \ , \end{aligned}$$

and leaving g inert.

We will be concerned with infinitesimal gauge transformations. To this end, let parametrise $\lambda = e^{\omega}$ and consider χ and $\bar{\chi}$ as infinitesimal parameters. The infinitesimal gauge transformations now read:

$$\begin{aligned} \delta g &= [g,\omega] \\ \delta\psi &= g^{-1}\chi g + [\psi,\omega] \\ \delta\bar{\psi} &= g\bar{\chi}g^{-1} + [\bar{\psi},\omega] \ . \end{aligned}$$

Since the bosonic field g is inert under the fermionic symmetry, the gauging of the bosonic part of the action is simply the gauged WZW model given by (I.4.5). We therefore focus on the fermionic part of the action (25).

Because of factorisation, we will consider each term separately. We will apply the Noether method to the action:

$$S^{(0)} = \int_\Sigma \left\langle \psi \, , \bar{\nabla} \psi \right\rangle \ ,$$

whose variation under an infinitesimal gauge transformation reads

$$\delta S^{(0)} = 2 \int \left\langle \bar{\partial}\omega \, , g\psi^2 g^{-1} \right\rangle - \left\langle \bar{\partial}\chi \, , g\psi g^{-1} \right\rangle \ .$$

We introduce at this point a bosonic gauge field $\bar{A}$ and a fermionic gauge field $\bar{\sigma}$, a $(0,1)$ form and a $(\frac{1}{2},1)$-form respectively, whose variations under an infinitesimal gauge transformation are given by

$$\begin{aligned} \delta \bar{A} &= \bar{\partial}\omega + [\bar{A}, \omega] \ , \\ \delta \bar{\sigma} &= \bar{\partial}\chi + [\bar{A}, \chi] + [\bar{\sigma}, \omega] \ . \end{aligned}$$

Following the Noether procedure we construct the first order correction to the $S^{(0)}$ to be equal to

$$S^{(1)} = -2 \int \left\langle \bar{A} \, , g\psi^2 g^{-1} \right\rangle - \left\langle \bar{\sigma} \, , g\psi g^{-1} \right\rangle \ .$$

This will cancel the two terms in $\delta S^{(0)}$ but will yield further

$$\delta(S^{(0)} + S^{(1)}) = 2 \int \left\langle \bar{\sigma} \, , \chi \right\rangle \ . \tag{41}$$

Now we face an obstruction, because one can easily see that there are no further local terms which can be added whose infinitesimal gauge transformation would cancel (41). One way around this problem is to parametrise the fermionic gauge field $\bar{\sigma}$ in terms of a new fermionic field η, a $(\frac{1}{2},0)$-form, such that

$$\bar{\sigma} = \bar{\partial}_{\bar{A}}\eta = \bar{\partial}\eta[\bar{A}, \eta] \ , \tag{42}$$

where we have introduced the covariant derivative $\bar{\partial}_{\bar{A}} = \bar{\partial} + [\bar{A}, -]$. The infinitesimal gauge transformation of η is fixed by that of σ to be:

$$\delta\eta = \chi + [\eta, \omega] \ .$$

Then we can add the following term to the action:

$$S^{(2)} = -\int \langle \bar{\sigma} \, , \eta \rangle \ \ .$$

The resulting action $S = S^{(0)} + S^{(1)} + S^{(2)}$ is indeed gauge invariant. In fact, it will be convenient to slightly modify this action so that (42) will appear naturally as an equation of motion. In that case the gauge invariant fermionic action will take the form

$$S = \int \left\langle g\psi g^{-1} \, , \bar{\partial}_A (g\psi g^{-1}) \right\rangle + 2 \left\langle \bar{\sigma} \, , g\psi g^{-1} \right\rangle - 2 \langle \bar{\sigma} \, , \eta \rangle - \left\langle \eta \, , \bar{\partial}_A \eta \right\rangle \ \ .$$

One can proceed in a similar fashion and gauge the other fermionic term in (25), introducing the corresponding gauge fields, A, σ and $\bar{\eta}$ (with $\sigma = \partial_A \bar{\eta}$), which transform like

$$\begin{aligned} \delta A &= \partial_A \omega \ , \\ \delta \sigma &= \partial_A \bar{\chi} + [\sigma, \omega] \ , \\ \delta \bar{\eta} &= \partial \bar{\chi} + [\bar{\eta}, \omega] \ . \end{aligned}$$

Then the full action of the gauged WZW model will be given by

$$\begin{aligned} I = {} & I_B[g, A, \bar{A}] \\ & - \int_\Sigma \left\langle g\psi g^{-1} \, , \bar{\partial}_{\bar{A}} (g\psi g^{-1}) \right\rangle + \left\langle g^{-1}\bar{\psi} g \, , \partial_A (g^{-1}\bar{\psi} g) \right\rangle \\ & + \int_\Sigma \left\langle \eta \, , \bar{\partial}_{\bar{A}} \eta \right\rangle + \langle \bar{\eta} \, , \partial_A \bar{\eta} \rangle \\ & - 2 \int_\Sigma \left\langle \sigma \, , g^{-1}\bar{\psi} g \right\rangle + \left\langle \bar{\sigma} \, , g\psi g^{-1} \right\rangle - \langle \sigma \, , \bar{\eta} \rangle - \langle \bar{\sigma} \, , \eta \rangle \ \ , \end{aligned} \tag{43}$$

or, more compactly, by:

$$\begin{aligned} I = {} & I_B[g, A, \bar{A}] - \int_\Sigma \left\langle g(\psi - \eta) g^{-1} \, , \bar{\partial}_{\bar{A}} g(\psi - \eta) g^{-1} \right\rangle \\ & - \int_\Sigma \left\langle g^{-1}(\bar{\psi} - \bar{\eta}) g \, , \partial_A g^{-1}(\bar{\psi} - \bar{\eta}) g \right\rangle \ \ . \end{aligned}$$

6.2 Witten's action from (8)

Let us now start from the gauged SWZW action (8) in superfields, which we recollect here for convenience:

$$I[\mathbb{G},\mathbb{A},\bar{\mathbb{A}}] = I[\mathbb{G}] - 2\int_{\Sigma_S} \left\langle \mathbb{J}\,,\bar{\mathbb{A}}\right\rangle + \left\langle \mathbb{A}\,,\bar{\mathbb{J}}\right\rangle - \left\langle \mathbb{A}\,,\bar{\mathbb{A}}\right\rangle + \left\langle \mathbb{A}\,,\mathbb{G}^{-1}\bar{\mathbb{A}}\mathbb{G}\right\rangle\ . \quad (44)$$

In the previous section we have seen how $I[\mathbb{G}]$ breaks down into components, yielding $I[g,\psi,\bar{\psi},a]$. The general expressions of the conserved supercurrents $\mathbb{J}$ and $\bar{\mathbb{J}}$ read

$$\begin{aligned}
\mathbb{J} &= -g\psi g - \theta\left(\partial g g^{-1} - g\psi^2 g^{-1}\right) - \bar{\theta} g a g^{-1} - \theta\bar{\theta}\left(\nabla\bar{\psi} - g[a,\psi]g^{-1}\right)\ ,\\
\bar{\mathbb{J}} &= g^{-1}\bar{\psi} g - \theta a + \bar{\theta}\left(g^{-1}\bar{\partial} g - g^{-1}\bar{\psi}^2 g\right) - \theta\bar{\theta}\left(\bar{\nabla}\psi + [a, g^{-1}\bar{\psi} g]\right)\ .
\end{aligned}$$

Finally, we parametrise the gauge superfields $\mathbb{A}$ and $\bar{\mathbb{A}}$ as follows:

$$\begin{aligned}
\mathbb{A} &= \rho + \theta A + \bar{\theta}\bar{B} + \theta\bar{\theta}\lambda\ ,\\
\bar{\mathbb{A}} &= \bar{\rho} + \theta B + \bar{\theta}\bar{A} + \theta\bar{\theta}\bar{\lambda}\ ,
\end{aligned}$$

where the bosonic components A, $\bar{A}$ correspond to the gauge fields in the bosonic case, as we will see in a moment. These new fields are not all independent, and one can see this in many ways. One way is to consider the equations of motion of the gauged action, one of which turns out to be the zero curvature condition for the gauge superfield. If we break this equation into components we obtain several relations between the above gauge field components. Here though we will follow a different approach.

After some tedious algebra, which includes solving the equation of motion for the field a, the gauged supersymmetric action can be written as

$$\begin{aligned}
I[\mathbb{G},\mathbb{A},\bar{\mathbb{A}}] = I_B[g, A, \bar{A}]&\\
+\int -\left\langle \psi\,,\bar{\nabla}_{\bar{\mathcal{A}}}\psi\right\rangle - \left\langle \bar{\psi}\,,\nabla_{\mathcal{A}}\bar{\psi}\right\rangle &+2\left\langle \nabla_{\mathcal{A}}\bar{\psi}\,,\bar{\rho}\right\rangle + 2\left\langle \rho\,,\bar{\nabla}_{\bar{\mathcal{A}}}\psi\right\rangle\\
+\int 2\left\langle \rho + g(\psi-\rho)g^{-1}\,,\bar{\lambda}\right\rangle &+2\left\langle \lambda\,,\bar{\rho} - g^{-1}(\bar{\psi}+\bar{\rho})g\right\rangle\\
+\int\Big(-\left\langle B+\bar{B}\,,B+\bar{B}\right\rangle &-2\left\langle \bar{B}\,,[g^{-1}(\bar{\psi}+\bar{\rho})g,\rho]\right\rangle\\
+\,2\left\langle B\,,[g(\psi-\rho)g^{-1},\bar{\rho}]\right\rangle &+2\left\langle \rho^2\,,g^{-1}\bar{\rho}^2 g\right\rangle\Big)\ ,
\end{aligned} \quad (45)$$

where we have introduced the following covariant derivatives

$$\nabla_{\mathcal{A}} \equiv \mathrm{ad}_g \circ \partial_{\mathcal{A}} \circ \mathrm{ad}_g^{-1} = \partial + [-\partial g g^{-1} + g\mathcal{A}g^{-1}, -] \; ,$$
$$\bar{\nabla}_{\bar{\mathcal{A}}} \equiv \mathrm{ad}_g^{-1} \circ \bar{\partial}_{\bar{\mathcal{A}}} \circ \mathrm{ad}_g = \bar{\partial} + [g^{-1}\bar{\partial} g + g^{-1}\bar{\mathcal{A}}g, -] \; ,$$

where $\mathcal{A} = A + \rho^2$ and similarly for $\bar{\mathcal{A}}$.

The first term in (45) is nothing but the bosonic gauged WZW action (I.4.5) with basic fields g and the gauge fields A and $\bar{A}$. In fact, it is easy to see that if we set the fermions to zero in $I[\mathbb{G}, \mathbb{A}, \bar{\mathbb{A}}]$ the gauged WZW action reduces to $I_B[g, A, \bar{A}]$ — the bosonic combination $B + \bar{B}$ also appears but it is trivially eliminated by its trivial equations of motion. Hence the A, $\bar{A}$ components of the gauge superfields correspond indeed to the gauge fields from the bosonic case, at least with fermions put to zero.

The second line in (45) can be rearranged to give (up to an overall sign)

$$\left\langle (\psi - \rho) \, , \bar{\nabla}_{\bar{\mathcal{A}}}(\psi - \rho) \right\rangle + \left\langle (\bar{\psi} + \bar{\rho}) \, , \nabla_{\mathcal{A}}(\bar{\psi} + \bar{\rho}) \right\rangle - \left\langle \rho \, , \bar{\nabla}_{\bar{\mathcal{A}}}\rho \right\rangle - \langle \bar{\rho} \, , \nabla_{\mathcal{A}}\bar{\rho} \rangle \;\; ,$$

which would look like a difference of two kinds of fermionic terms, were it not for the g-dependent covariant derivatives $\nabla_{\mathcal{A}}$ and $\bar{\nabla}_{\bar{\mathcal{A}}}$. This fact can be remedied by suitably rewriting these terms, together with the gauged bosonic action:

$$\begin{aligned} & I_B[g, A, \bar{A}] + \int \left\langle \rho \, , \bar{\nabla}_{\bar{A}}\rho \right\rangle + \langle \bar{\rho} \, , \nabla_{\mathcal{A}}\bar{\rho} \rangle = \\ & I_B[g, \mathcal{A}, \bar{\mathcal{A}}] + \int \left\langle \rho \, , \bar{\partial}_{\bar{A}}\rho \right\rangle + \langle \bar{\rho} \, , \partial_{\mathcal{A}}\bar{\rho} \rangle + 2 \left\langle \rho^2 \, , \bar{\rho}^2 \right\rangle - 2 \left\langle \rho^2 \, , g^{-1}\bar{\rho}^2 g \right\rangle \;\; . \end{aligned}$$

This gives us back the bosonic action I_B, written in terms of modified bosonic gauge fields $\mathcal{A}$ and $\bar{\mathcal{A}}$, whereas the resulting fermionic terms describe $\mathfrak{h}$-fermions minimally coupled to the $\mathfrak{h}$-valued gauge fields $\mathcal{A}$, $\bar{\mathcal{A}}$ through the covariant derivatives. The full action can be written as

$$\begin{aligned} I[\mathbb{G}, \mathbb{A}, \bar{\mathbb{A}}] = {} & I_B[g, \mathcal{A}, \bar{\mathcal{A}}] \\ & + \int - \left\langle (\psi - \rho) \, , \bar{\nabla}_{\bar{\mathcal{A}}}(\psi - \rho) \right\rangle - \left\langle (\bar{\psi} + \bar{\rho}) \, , \nabla_{\mathcal{A}}(\bar{\psi} + \bar{\rho}) \right\rangle \\ & + \int \left\langle \rho \, , \bar{\partial}_{\bar{A}}\rho \right\rangle + \langle \bar{\rho} \, , \partial_{\mathcal{A}}\bar{\rho} \rangle \\ & + \int 2 \left\langle \rho + g(\psi - \rho)g^{-1} \, , \bar{\lambda} \right\rangle + 2 \left\langle \lambda \, , \bar{\rho} - g^{-1}(\bar{\psi} + \bar{\rho})g \right\rangle \\ & + \int - \left\langle B + \bar{B} \, , B + \bar{B} \right\rangle - 2 \left\langle \bar{B} \, , [g^{-1}(\bar{\psi} + \bar{\rho})g, \rho] \right\rangle \\ & + 2 \left\langle B \, , [g(\psi - \rho)g^{-1}, \bar{\rho}] \right\rangle + 2 \left\langle \rho^2 \, , \bar{\rho}^2 \right\rangle \;\; , \end{aligned} \tag{46}$$

Notice that the λ, $\bar{\lambda}$ gauge fields have no kinetic terms (dynamics), but rather play the role of Lagrange multipliers, imposing the following constraint equations on the fermionic fields

$$\bar{\rho} = g^{-1}(\bar{\psi} + \bar{\rho})g|_{\mathfrak{h}} \ , \qquad -\rho = g(\psi - \rho)g^{-1}|_{\mathfrak{h}} \ , \tag{47}$$

where $|_{\mathfrak{h}}$ denotes the orthogonal projection $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}^{\perp} \to \mathfrak{h}$.

If we introduce these equations back in the gauged action and we solve the equation of motion for the field $B + \bar{B}$ we obtain

$$B + \bar{B} = -[\rho, \bar{\rho}] \ ,$$

which will cancel the quartic term in ρ, $\bar{\rho}$. At this moment we are left only with the bosonic gauged action and the fermionic terms, corresponding roughly to minimally coupled $\mathfrak{g}$- and $\mathfrak{h}$-fermions. In order to see what the equations of motion (47) impose on the $\mathfrak{g}$-fermions, we will decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$:

$$\begin{aligned}
\Big\langle (\psi - \rho) \, , \bar{\nabla}_{\bar{\mathcal{A}}}(\psi - \rho) \Big\rangle_{\mathfrak{g}} &= \Big\langle g(\psi - \rho)g^{-1} \, , \bar{\partial}_{\bar{\mathcal{A}}} g(\psi - \rho)g^{-1} \Big\rangle_{\mathfrak{g}} \\
&= \Big\langle \rho \, , \bar{\partial}_{\bar{\mathcal{A}}} \rho \Big\rangle_{\mathfrak{h}} + \Big\langle g(\psi - \rho)g^{-1} \, , \bar{\partial}_{\bar{\mathcal{A}}} g(\psi - \rho)g^{-1} \Big\rangle_{\mathfrak{g}/\mathfrak{h}} \\
\Big\langle (\bar{\psi} + \bar{\rho}) \, , \nabla_{\mathcal{A}}(\bar{\psi} + \bar{\rho}) \Big\rangle_{\mathfrak{g}} &= \Big\langle g^{-1}(\bar{\psi} + \bar{\rho})g \, , \partial_{\mathcal{A}} g^{-1}(\bar{\psi} + \bar{\rho})g \Big\rangle_{\mathfrak{g}} \\
&= \langle \bar{\rho} \, , \partial_{\mathcal{A}} \bar{\rho} \rangle_{\mathfrak{h}} + \Big\langle g^{-1}(\bar{\psi} + \bar{\rho})g \, , \partial_{\mathcal{A}} g^{-1}(\bar{\psi} + \bar{\rho})g \Big\rangle_{\mathfrak{g}/\mathfrak{h}} \ ,
\end{aligned}$$

where we use the suggestive notation $\mathfrak{g}/\mathfrak{h}$ to mean $\mathfrak{h}^{\perp}$.

The first of the two terms in each RHS clearly cancel the similar terms in (46) whereas the remaining terms can be rewritten in a more compact form if we define new $\mathfrak{g}/\mathfrak{h}$-valued fermionic fields

$$\Psi_{\mathfrak{g}/\mathfrak{h}} \equiv g(\psi - \rho)g^{-1}|_{\mathfrak{g}/\mathfrak{h}} \qquad \text{and} \qquad \bar{\Psi}_{\mathfrak{g}/\mathfrak{h}} \equiv g^{-1}(\bar{\psi} + \bar{\rho})g|_{\mathfrak{g}/\mathfrak{h}} \ .$$

Putting all this together, we arrive at the action

$$I[\mathbb{G}, \mathbb{A}, \bar{\mathbb{A}}] = I_B[g, \mathcal{A}, \bar{\mathcal{A}}] - \int \Big\langle \Psi_{\mathfrak{g}/\mathfrak{h}} \, , \bar{\partial}_{\bar{\mathcal{A}}} \Psi_{\mathfrak{g}/\mathfrak{h}} \Big\rangle + \Big\langle \bar{\Psi}_{\mathfrak{g}/\mathfrak{h}} \, , \partial_{\mathcal{A}} \bar{\Psi}_{\mathfrak{g}/\mathfrak{h}} \Big\rangle \ , \tag{48}$$

which was introduced by Witten in [29].

Notice nevertheless that the basic fields entering in this action are not the ones that we would have naively expected. Indeed, the $\mathfrak{g}/\mathfrak{h}$-fermions do not coincide with the $\mathfrak{g}/\mathfrak{h}$ subset of the original $\mathfrak{g}$-fermions, rather they differ by

a shift and an axial gauge transformation from these. Also, the bosonic gauge fields $\mathcal{A}$, $\bar{\mathcal{A}}$ differ from the gauge fields that appear in the bosonic gauged WZW action by a shift quadratic in the fermionic gauge fields ρ, $\bar{\rho}$.

6.3 Equivalence with $I[g, \psi, A, \sigma, \rho]$

We have now two classical gauged WZW actions: the action (43) obtained by gauging (à la Noether) the component SWZW action (25), and the Witten action (48) obtained by breaking down the superfield gauged SWZW action (8) into components. The natural question arises whether they are indeed equivalent, and in this subsection we show how to relate the two.

In order to do this we go back to the gauged action (46) and we make a change of variables, replacing the fields λ, $\bar{\lambda}$ with the following combinations:

$$\tau \equiv \lambda + [\bar{B}, \rho] \ , \qquad \bar{\tau} \equiv \bar{\lambda} - [B, \bar{\rho}] \ .$$

If we introduce this back in the action, and we solve the algebraic equation of motion for $B + \bar{B}$ we obtain:

$$\begin{aligned} I[\mathbb{G}, \mathbb{A}, \bar{\mathbb{A}}] = I_B[g, \mathcal{A}, \bar{\mathcal{A}}] \\ & - \int \left\langle (\psi - \rho) \, , \bar{\nabla}_{\mathcal{A}}(\psi - \rho) \right\rangle + \left\langle (\bar{\psi} + \bar{\rho}) \, , \nabla_{\mathcal{A}}(\bar{\psi} + \bar{\rho}) \right\rangle \\ & + \int \left\langle \rho \, , \bar{\partial}_{\bar{\mathcal{A}}} \rho \right\rangle + \langle \bar{\rho} \, , \partial_{\mathcal{A}} \bar{\rho} \rangle + 2 \, \langle \tau \, , \bar{\rho} \rangle + 2 \, \langle \rho \, , \bar{\tau} \rangle \\ & - 2 \int \left\langle \tau \, , g^{-1}(\bar{\psi} + \bar{\rho}) g \right\rangle + \left\langle \bar{\tau} \, , g(\psi - \rho) g^{-1} \right\rangle \ . \end{aligned} \tag{49}$$

It is easy now to compare this action with the one obtained in the previous section and show their equivalence. Indeed, the following dictionary provides the equivalence:

(43)	(49)	(43)	(49)	(43)	(49)	(43)	(49)
ψ	$\psi - \rho$	A	$\mathcal{A}$	σ	τ	η	$-\rho$
$\bar{\psi}$	$\bar{\psi} + \bar{\rho}$	$\bar{A}$	$\bar{\mathcal{A}}$	$\bar{\sigma}$	$\bar{\tau}$	$\bar{\eta}$	$\bar{\rho}$

In summary, gauging the component SWZW model yields the same theory as the gauging the superfield SWZW model, and both theories are equivalent to the one written down by Witten in [29].

7 Nonreductive Kazama-Suzuki models

Under certain circumstances the N=1 coset theory admits an extra supersymmetry giving rise to an N=2 coset. For $\mathfrak{g}$ a reductive Lie algebra this is the celebrated Kazama-Suzuki construction [14] (see also [25]). The purpose of this section is to extend this construction to the case of self-dual Lie algebras.

7.1 N=1 coset construction in components

In Section 5 we have studied in considerable detail the expression in components of the N=1 affine algebra $\widehat{\mathfrak{g}}_{N=1}$, both in the coupled (31)-(33) and in the decoupled (37)-(39) basis, and the ones of the two generators of the N=1 superconformal algebra (in terms of these currents). Also, in Section 4 we have considered the N=1 coset construction, written in terms of superfields. In order to proceed further and investigate the existence of an N=2 extension to this N=1 superconformal algebra we need to start with the N=1 coset theory written in components. It is convenient to use a modified decoupled basis. If

$$\mathbb{J}_i(Z) = \Psi_i(z) + \theta I_i(z) \ ,$$

then we define a modified bosonic current as follows:

$$\widetilde{J}_i(z) \equiv I_i(z) - \tfrac{1}{2}\Omega^{jl} f_{ij}{}^k (\Psi_k \Psi_l)(z) \ .$$

Notice that this differs from $J_i(z)$ defined in (36), but it is nevertheless decoupled from the $\mathfrak{h}$-fermions:

$$\widetilde{J}_i(z)\Psi_j(w) = \text{reg} \ .$$

These currents define a realisation of an affine Lie algebra $\widehat{\mathfrak{h}}$

$$\widetilde{J}_i(z)\widetilde{J}_j(w) = \frac{\Omega_{ij} - \frac{1}{2}\kappa^{\mathfrak{h}}_{ij}}{(z-w)^2} + \frac{f_{ij}{}^k \widetilde{J}_k(w)}{z-w} + \text{reg} \ ,$$

where the shift in the metric is now proportional to $\kappa^{\mathfrak{h}}$, the Killing form for $\mathfrak{h}$.

In this basis, the $\mathfrak{h}$ N=1 Virasoro generators read:

$$\begin{aligned}
\mathsf{T}_{\mathfrak{h}}(z) &= \tfrac{1}{2}\Omega^{ij}(\widetilde{J}_i\widetilde{J}_j)(z) + \tfrac{1}{2}\Omega^{ij}(\partial\Psi_i\Psi_j)(z) \ , \\
\mathsf{G}_{\mathfrak{h}}(z) &= \Omega^{ij}(\widetilde{J}_i\Psi_j)(z) - \tfrac{1}{6} f^{ijk}(\Psi_i(\Psi_j\Psi_k))(z) \ .
\end{aligned}$$

The N=1 coset theory generated by $\mathsf{G} \equiv \mathsf{G}_{\mathfrak{g}/\mathfrak{h}} = \mathsf{G}_{\mathfrak{g}} - \mathsf{G}_{\mathfrak{h}}$ and $\mathsf{T} \equiv \mathsf{T}_{\mathfrak{g}/\mathfrak{h}} = \mathsf{T}_{\mathfrak{g}} - \mathsf{T}_{\mathfrak{h}}$ satisfies the algebra

$$\begin{aligned}
\mathsf{T}(z)\mathsf{T}(w) &= \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2\mathsf{T}(w)}{(z-w)^2} + \frac{\partial\mathsf{T}(w)}{z-w} + \text{reg} \ , \\
\mathsf{T}(z)\mathsf{G}(w) &= \frac{\frac{3}{2}\mathsf{G}(w)}{(z-w)^2} + \frac{\partial\mathsf{G}(w)}{z-w} + \text{reg} \ , \\
\mathsf{G}(z)\mathsf{G}(w) &= \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2\mathsf{T}(w)}{z-w} + \text{reg} \ ,
\end{aligned}$$

with the central charge c given by (15).

7.2 N=2 superconformal cosets

Now we want to solve the following problem: We want to determine the conditions under which these N=1 theories possess an N=2 superconformal symmetry. In other words, we want to determine the conditions under which we can define two new operators, say G^2 and J, such that $(\mathsf{T}, \mathsf{G}^+, \mathsf{G}^-, \mathsf{J})$ satisfy an N=2 superconformal algebra, where

$$\mathsf{G}^+ = \tfrac{1}{2}(\mathsf{G}^1 + i\mathsf{G}^2) \ , \qquad \mathsf{G}^- = \tfrac{1}{2}(\mathsf{G}^1 - i\mathsf{G}^2) \ ,$$

whereas T and $\mathsf{G}^1 = \mathsf{G}$ are the N=1 coset generators introduced in the last paragraph.

In order to do this we will make use of the following characterisation of the N=2 Virasoro algebra, proven independently in [3] and [9]. The result states that the minimal data necessary to guarantee the existence of an N=2 superconformal algebra consists of two fields $\mathsf{G}^\pm(z)$ satisfying

$$\begin{aligned}
\mathsf{G}^\pm(z)\mathsf{G}^\pm(w) &= \text{reg} \ , \\
\mathsf{G}^+(z)\mathsf{G}^-(w) &= \frac{\frac{1}{3}c}{(z-w)^3} + \frac{\mathsf{J}(w)}{(z-w)^2} + \frac{\mathsf{T}(w) + \frac{1}{2}\partial\mathsf{J}(w)}{z-w} + \text{reg} \ ,
\end{aligned}$$

which defines the central charge c, and the operators J and T; and also such that

$$\mathsf{J}(z)\mathsf{G}^\pm(w) = \frac{\pm\mathsf{G}^\pm(w)}{z-w} + \text{reg} \ . \tag{50}$$

In other words, provided the above OPEs are satisfied, $(\mathsf{T}, \mathsf{G}^\pm, \mathsf{J})$ will satisfy an N=2 superconformal algebra with central charge c.

In our case G^1 has a rather simple expression. We split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, as usual and we introduce bases $\{X_i\}$ and $\{X_\alpha\}$ for $\mathfrak{h}$ and $\mathfrak{h}^\perp$ respectively. Of course, as vector spaces (and even as $\mathfrak{h}$-modules) $\mathfrak{h}^\perp \cong \mathfrak{g}/\mathfrak{h}$, and we will on occasion allow ourselves to use $\mathfrak{g}/\mathfrak{h}$ as a shorthand for the subspace $\mathfrak{h}^\perp \in \mathfrak{g}$. Then G^1 can be written in terms of the $\mathfrak{g}/\mathfrak{h}$-fields:

$$\mathsf{G}^1 = \Omega^{\alpha\beta}(J_\alpha \Psi_\beta) - \tfrac{1}{6} f^{\alpha\beta\gamma}(\Psi_\alpha(\Psi_\beta\Psi_\gamma)) \ .$$

From the above discussion it follows that we need another fermionic field $\mathsf{G}^2(z)$ such that

$$\begin{aligned}
\mathsf{G}^1(z)\mathsf{G}^2(w) &= \frac{2i\mathsf{J}(w)}{(z-w)^2} + \frac{i\partial\mathsf{J}(w)}{z-w} + \text{reg} \ , \\
\mathsf{G}^2(z)\mathsf{G}^2(w) &= \frac{\frac{3}{2}c}{(z-w)^3} + \frac{2\mathsf{T}(w)}{z-w} + \text{reg} \ ,
\end{aligned}$$

and then, having determined the $U(1)$ current $\mathsf{J}(z)$ from the first OPE, we will have to impose the additional OPEs between J and $\mathsf{G}^{1,2}$. We will find it convenient to rewrite the above two OPEs using the ϵ-symbol (with $\epsilon_{12} = 1 = -\epsilon_{21}$):

$$\mathsf{G}^i(z)\mathsf{G}^j(w) = \frac{\frac{2}{3}c\delta_{ij}}{(z-w)^3} + \frac{2i\mathsf{J}(w)\epsilon_{ij}}{(z-w)^2} + \frac{2\mathsf{T}(w)\delta_{ij} + i\partial\mathsf{J}(w)\epsilon_{ij}}{z-w} + \text{reg} \ . \quad (51)$$

We start with the following Ansatz for G^2:

$$\mathsf{G}^2(z) = A^{\alpha\beta}(J_\alpha\Psi_\beta)(z) + \tfrac{1}{6}B^{\alpha\beta\gamma}(\Psi_\alpha(\Psi_\beta\Psi_\gamma))(z) + C^\alpha\partial\Psi_\alpha(z) \ ,$$

where A, B, C are still to be determined. By imposing (51) and (50) and after some lengthy computations we obtain a set of necessary and sufficient conditions which after a lot of effort can be reduced to the following:

(i) $C^\alpha = 0$.

(ii) The matrix $(A^{\alpha\beta})$ defines an $\mathfrak{h}$-invariant almost complex structure on $\mathfrak{g}/\mathfrak{h}$:

$$A^{\alpha\beta}A^{\gamma\delta}\Omega_{\beta\gamma} = -\Omega^{\alpha\delta} \ , \qquad A^{\alpha\beta} = -A^{\beta\alpha} \ , \quad (52)$$

$$A^{\alpha\beta}f_\beta{}^{\gamma i} = A^{\gamma\beta}f_\beta{}^{\alpha i} \ . \quad (53)$$

One can easily see this by defining a map $A : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$, with $A \cdot X_\alpha \equiv A^\beta{}_\alpha X_\beta$, where $A^\beta{}_\alpha = A^{\beta\gamma}\Omega_{\gamma\alpha}$; then the first equation in (52) is equivalent with $A^2 = -\mathbf{1}$. On the other hand the antisymmetry of $(A^{\alpha\beta})$ tells us that the complex structure is compatible with the metric on $\mathfrak{g}/\mathfrak{h}$

$$\langle A \cdot X_\alpha \, , A \cdot X_\beta \rangle = \langle X_\alpha \, , X_\beta \rangle \ .$$

Finally, the relation (53) states the $\mathfrak{h}$-invariance of the complex structure on $\mathfrak{g}/\mathfrak{h}$. Indeed, one can define an action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ by understanding $\mathfrak{g}/\mathfrak{h}$ as $\mathfrak{h}^\perp \subset \mathfrak{g}$ and using the Lie brackets, and (53) can be expressed as the fact that A commutes with the above $\mathfrak{h}$ action:

$$A \cdot [X_i, X_\alpha] = [X_i, A \cdot X_\alpha] \; .$$

(iii) The coefficient of the cubic term in the expression on G^2 is given by

$$B^{\mu\nu\rho} = A^{\mu\alpha} A^{\nu\beta} A^{\rho\gamma} f_{\alpha\beta\gamma} \; .$$

(iv) Finally, the last condition that we obtain

$$f^{\mu\nu\rho} = A^{\mu\alpha} A^{\nu\beta} f_{\alpha\beta}{}^{\rho} + A^{\nu\alpha} A^{\rho\beta} f_{\alpha\beta}{}^{\mu} + A^{\rho\alpha} A^{\mu\beta} f_{\alpha\beta}{}^{\nu} \; ,$$

may seem formidable at first sight, but it is in fact equivalent to the vanishing of the Nijenhuis tensor associated to the complex structure A:

$$N(X,Y) \equiv [X,Y] - [AX,AY] + A[X,AY] + A[AX,Y] \; .$$

One can give an alternative interpretation to the last two conditions. For this, let us introduce the following projection operators

$$(P^\pm)^\alpha_\beta = \tfrac{1}{2}(\mathbf{1}^\alpha_\beta \pm \frac{1}{i} A^\alpha_\beta) \; ,$$

which allows us to split the complexification $\mathfrak{t} = (\mathfrak{h}^\perp)^{\mathbb{C}}$ into subspaces $\mathfrak{t}_+$ and $\mathfrak{t}_-$ defined as the image of the projectors P^+ and P^- respectively. Introducing bases

$$\{X^\pm_\alpha = (P^\pm)^\beta_\alpha X_\beta\}$$

for $\mathfrak{t}_\pm$ respectively, we can then show that (52) implies that $\mathfrak{t}_\pm$ are (maximally) isotropic, whereas *(iii)* and *(iv)* are equivalent to:

$$[X^\pm_\alpha, X^\pm_\beta] = \tfrac{1}{2} i (f_{\alpha\beta}{}^{\gamma} \pm i B_{\alpha\beta}{}^{\gamma}) X^\pm_\gamma \; .$$

This means that $\mathfrak{t}$ admits a decomposition $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$ into subspaces which close under the Lie brackets:

$$[\mathfrak{t}_+, \mathfrak{t}_+] \subset \mathfrak{t}_+ \; , \qquad [\mathfrak{t}_-, \mathfrak{t}_-] \subset \mathfrak{t}_- \; ,$$

which re-states the fact that the complex structure in $\mathfrak{g}/\mathfrak{h}$ is integrable.

Notice that if $\mathfrak{h} = 0$, then the condition of $\mathfrak{h}$-invariance would be trivially satisfied, and the remaining conditions are precisely the ones in [17] (see also

[5]). In that case, $\mathfrak{t} = \mathfrak{g}^{\mathbb{C}}$ and $(\mathfrak{t}, \mathfrak{t}_+, \mathfrak{t}_-)$ would be a Manin triple. In the more general case, what we have is that $\mathfrak{t}_\pm$ are isotropic subalgebras of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}_+ \oplus \mathfrak{t}_-$. In the reductive case, Getzler [9] has shown that there is indeed an honest Manin triple underlying the coset construction, with "double" given by $\mathfrak{g} \oplus (-\mathfrak{h})$, where $-\mathfrak{h}$ is $\mathfrak{h}$ with the opposite metric; although the precise relation between Getzler's Manin triple and the KS construction will be fully elucidated elsewhere.

In summary, provided $\mathfrak{g}/\mathfrak{h}$ has an $\mathfrak{h}$-invariant metric with a compatible, integrable, $\mathfrak{h}$-invariant complex structure, the corresponding N=1 supersymmetric coset possesses an extended N=2 superconformal symmetry. This N=2 Virasoro algebra is generated by T, G^1, and in addition the two generators

$$\mathsf{G}^2 = \Omega^{\alpha\beta}(J_\alpha \widetilde{\Psi}_\beta) - \tfrac{1}{6} f^{\alpha\beta\gamma}(\widetilde{\Psi}_\alpha(\widetilde{\Psi}_\beta \widetilde{\Psi}_\gamma)) \ , \tag{54}$$

where $\widetilde{\Psi}_\alpha = A^\beta{}_\alpha \Psi_\beta$; and the $U(1)$ current whose expression turns out to be

$$2i\mathsf{J} = A^{\alpha\beta}(\Psi_\alpha \Psi_\beta) - A^{\gamma\delta} f_{\gamma\delta}{}^c I_c \ , \tag{55}$$

whereas the central charge is given by (15). Notice that it is not possible to assemble J and G^2 given by (54) and (55) above into a superfield depending polynomially in the original superfields $\mathbb{J}_a$, whence the need to work in components.

7.3 The BRST invariance of the N=2 generators

We have shown above that the gauged supersymmetric WZW model describes the N=1 coset theory. It thus makes sense that any extended symmetry of the N=1 Virasoro algebra which the coset theory admits, must be already present (maybe up to BRST-exact terms) among the BRST-invariant fields in the WZW model. Therefore we expect that the N=2 extension, whenever it exists, must be BRST-invariant or, in this case, since they don't involve the ghosts, actually gauge invariant. Since J and G^1 generate the rest of the N=2 Virasoro algebra, and G^1 is already BRST-invariant, all we need to show is that J is BRST-invariant.

For this we have to first work out the expression of the BRST current in components. A convenient parametrisation of the ghost superfields is given by:

$$\mathbb{B}_i(Z) = \beta_i(z) - \theta b_i(z) \qquad \text{and} \qquad \mathbb{C}^i(Z) = -c^i(z) - \theta\gamma^i(z) \ ,$$

where (β_i, γ^i) are bosonic fields with weights $(\frac{1}{2}, \frac{1}{2})$ and (b_i, c^i) are fermionic fields with weights $(1,0)$. Their OPEs can be read from the ones of $(\mathbb{B}_i, \mathbb{C}^i)$:

$$b_i(z) c^j(w) = \frac{\delta_i^j}{z-w} \qquad \text{and} \qquad \beta_i(z) \gamma^j(w) = \frac{\delta_i^j}{z-w} \ .$$

The BRST current $q(z)$ is the θ-component of the superfield in (22):

$$q(z) = (I_i + \tilde{I}_i) c^i - (\psi_i + \tilde{\psi}_i) \gamma^i + f_{ij}{}^k \beta_k c^i \gamma^j - \tfrac{1}{2} f_{ij}{}^k b_k c^i c^j \ ,$$

and BRST transformations on fields are given by $d\phi = [q, \phi]_1$ in the notation of the Appendix.

It is now a simple matter to prove that the expression (55) for $\mathsf{J}(z)$ is BRST invariant. In fact, this follows trivially from the $\mathfrak{h}$-invariance of the complex structure. To this effect, notice that the BRST transformation of the coset fermions ψ_α is precisely an $\mathfrak{h}$-gauge rotation:

$$d\psi_\alpha = -f_{i\alpha}{}^\beta \psi_\beta c^i \ .$$

Hence any $\mathfrak{h}$-invariant tensor contracted with coset fermions is automatically BRST-invariant. This, together with the identity

$$A^{\alpha\beta} f_{\alpha\beta}{}^c f_{ci}{}^b = 0 \ ,$$

is enough to show that J is BRST-invariant. This proves that the gauged supersymmetric WZW model does provide a lagrangian realisation of the $N{=}2$ coset construction.

One might wonder whether in the same way that the $N{=}1$ coset theory is induced from a natural $N{=}1$ SCFT involving all three sectors in the gauged supersymmetric WZW model: $(\mathsf{G}_{\text{tot}}, \mathsf{T}_{\text{tot}})$, the same is true for the $N{=}2$ coset. In other words, *is there a natural BRST invariant* $N{=}2$ *SCFT involving the ghosts, extending* $(\mathsf{G}_{\text{tot}}, \mathsf{T}_{\text{tot}})$*, and which is BRST-cohomologous to the one generated by* $(\mathsf{J}, \mathsf{G}^1, \mathsf{G}^2, \mathsf{T})$*?*

Let us try to answer this question. Notice first of all that the ghost $N{=}1$ Virasoro algebra does extend to an $N{=}2$ with generators:

$$\mathsf{J}_{\text{gh}} \equiv \beta_i \gamma^i \qquad \text{and} \qquad \mathsf{G}^2_{\text{gh}} \equiv -i \left(b_i \gamma^i - \beta_i \partial c^i \right) \ ,$$

in addition to $\mathsf{G}^1_{\text{gh}} \equiv \mathsf{G}_{\text{gh}}$ and T_{gh}. We would therefore need to find a BRST-

exact $\mathsf{J}' = \mathsf{J}_{\mathrm{gh}} + \cdots$. It turns out that there is a unique such J':

$$\mathsf{J}' \equiv \beta_i \gamma^i + \Omega^{ij} \psi_i \tilde{\psi}_j = d\left(-\tfrac{1}{2}\Omega^{ij}\beta_i \psi_j^-\right) \ .$$

Let us define $\mathsf{J}_{\mathrm{tot}} \equiv \mathsf{J} + \mathsf{J}'$. Similarly let $\mathsf{G}^1_{\mathrm{tot}} \equiv \mathsf{G}_{\mathrm{tot}}$. It turns out that these generators do not form generally satisfy an N=2 superconformal algebra, but they do when $\mathfrak{h}$ is abelian! The passage from the total N=2 superconformal algebra to the coset one can be understood as the conformal field theoretical manifestation of the Poisson reduction of a Poisson Lie group. Details will appear elsewhere.

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A Supersymmetric OPE Technology

In this appendix we collect some formulas which are useful in the computations concerning superspace operator product expansions in meromorphic superconformal field theory. See [15] for the computer implementation of these formulas (and their N=2 extension).

Our superfields are functions $\Phi(Z)$ in a $(1|1)$-superspace whose points are denoted by $Z = (z, \theta)$, with z even and θ odd. The supercovariant derivative is given by $D = \frac{\partial}{\partial\theta} + \theta\partial$, where we use the abbreviation ∂ to mean $\frac{\partial}{\partial z}$. The supercovariant derivative obeys $D^2 = \partial$.

Given two points $Z_i = (z_i, \theta_i)$ for $i = 1, 2$, we define even and odd superintervals:

$$Z_{12} = \text{“}Z_1 - Z_2\text{”} \equiv z_1 - z_2 - \theta_1\theta_2 \qquad \text{and} \qquad \theta_{12} = \text{“}Z_{12}^{\frac{1}{2}}\text{”} \equiv \theta_1 - \theta_2 \ ,$$

where the notation $Z_{12}^{\frac{1}{2}}$ has been introduced for convenience. More generally, $Z_{12}^{n+\frac{1}{2}} = Z_{12}^n \theta_{12}$ for any $n \in \mathbb{Z}$.

There exists a supersymmetric analogue of the Cauchy residue calculus. Defining the measure $dZ \equiv \frac{dz}{2\pi i} d\theta$, and the contour integral $\oint_Z$ to refer both to the contour integral about z and the Berezin integral, Laurent expansions take the form

$$\Phi(Z_1) = \sum_{2r \in \mathbb{Z}} Z_{12}^{-r-\frac{1}{2}} \, \Phi^{(r)}(Z_2) \ ,$$

where

$$\Phi^{(r)}(Z_2) = \oint_{Z_2} dZ_1 \, Z_{12}^r \, \Phi(Z_1) \ .$$

And in particular, Taylor expansions are of the form

$$\Phi(Z_1) = \sum_{2r \in |\mathbb{Z}|} \frac{1}{\lfloor r \rfloor !} \, Z_{12}^r \, D^{2r} \Phi(Z_2) \ ,$$

where

$$\oint_{Z_2} dZ_1 \, Z_{12}^{-(r+\frac{1}{2})} \, \Phi(Z_1) = \frac{1}{\lfloor r \rfloor !} \, D^{2r} \Phi(Z_2) \qquad \text{for } 2r \in |\mathbb{Z}| \ ,$$

where $\lfloor r \rfloor$ denotes the greatest integer $\leq r$.

The superspace operator product expansion takes the form

$$\mathbb{A}(Z_1)\mathbb{B}(Z_2) = \sum_{2r \in \mathbb{Z}} Z_{12}^{-r} \, [\![\mathbb{A}, \mathbb{B}]\!]_r(Z_2) \ ,$$

where by definition,

$$[\![\mathbb{A}, \mathbb{B}]\!]_r(Z_2) = \oint_{Z_2} dZ_1 \, Z_{12}^{r-\frac{1}{2}} \, \mathbb{A}(Z_1)\mathbb{B}(Z_2) \ .$$

If $\mathbb{A}(Z) = \phi_A(z) + \theta\psi_A(z)$ and $\mathbb{B}(Z) = \phi_B(z) + \theta\psi_B(z)$, the brackets $[\![\mathbb{A}, \mathbb{B}]\!]_r$ can be written in terms of the similar brackets of the component fields as follows:

$$
\begin{aligned}
[\![\mathbb{A},\mathbb{B}]\!]_n(Z) &= [\phi_A,\phi_B]_n(z) + \theta\left([\psi_A,\phi_B]_n(z) + (-)^{|\mathbb{A}|}[\phi_A,\psi_B]_n(z)\right)\\
[\![\mathbb{A},\mathbb{B}]\!]_{n+\frac{1}{2}}(Z) &= [\psi_A,\phi_B]_{n+1}(z) - \theta\left(n[\phi_A,\phi_B]_n(z) + (-)^{|\mathbb{A}|}[\psi_A,\psi_B]_{n+1}(z)\right)
\end{aligned}
$$

for every $n \in \mathbb{Z}$, where the brackets $[-,-]_n$ are defined as usual by the ordinary operator product expansion:

$$
A(z)B(w) = \sum_n \frac{[A,B]_n(w)}{(z-w)^n} \ .
$$

Then either from the identities obeyed by the $[-,-]_n$ or working directly from the properties of the superspace operator product expansion, one can derive a set of "axioms" obeyed by the $[\![-,-]\!]_r$. These axioms encode the properties of identity, commutativity and associativity of the operator product expansion, as well as the properties of the normal ordered product and of the supercovariant derivative D. Since these two operations generate the operator algebra of the superconformal field theory starting from a set of generating fields, the above axioms allow us to compute *all* $[\![-,-]\!]_r$ knowing only the $[\![-,-]\!]_{r>0}$ of the generating fields.

It is convenient in what follows to separately discuss the brackets $[\![-,-]\!]_n$ and $[\![-,-]\!]_{n+\frac{1}{2}}$, where $n \in \mathbb{Z}$. We first have the identity axiom:

$$
[\![\mathbf{1},\mathbb{A}]\!]_r = \begin{cases} \mathbb{A}\ , & \text{for } r = 0 \\ 0\ , & \text{otherwise} \end{cases} \ ,
$$

where $\mathbf{1}$ is the identity.

We then have the action of the supercovariant derivative:

$$
\begin{aligned}
[\![D\mathbb{A},\mathbb{B}]\!]_n &= [\![\mathbb{A},\mathbb{B}]\!]_{n-\frac{1}{2}}\ ,\\
[\![D\mathbb{A},\mathbb{B}]\!]_{n+\frac{1}{2}} &= -n[\![\mathbb{A},\mathbb{B}]\!]_n\ ,\\
[\![\mathbb{A},D\mathbb{B}]\!]_n &= (-)^{|\mathbb{A}|}\left(D[\![\mathbb{A},\mathbb{B}]\!]_n - [\![\mathbb{A},\mathbb{B}]\!]_{n-\frac{1}{2}}\right)\ ,\\
[\![\mathbb{A},D\mathbb{B}]\!]_{n+\frac{1}{2}} &= -(-)^{|\mathbb{A}|}\left(D[\![\mathbb{A},\mathbb{B}]\!]_{n+\frac{1}{2}} + n[\![\mathbb{A},\mathbb{B}]\!]_n\right)\ .
\end{aligned}
$$

Iterating these relations we find the ones for ∂:

$$
\begin{aligned}
[\![\partial\mathbb{A},\mathbb{B}]\!]_n &= (1-n)[\![\mathbb{A},\mathbb{B}]\!]_{n-1}\ ,\\
[\![\partial\mathbb{A},\mathbb{B}]\!]_{n+\frac{1}{2}} &= -n[\![\mathbb{A},\mathbb{B}]\!]_{n-\frac{1}{2}}\ ,\\
[\![\mathbb{A},\partial\mathbb{B}]\!]_n &= \partial[\![\mathbb{A},\mathbb{B}]\!]_n + (n-1)[\![\mathbb{A},\mathbb{B}]\!]_{n-1}\ ,\\
[\![\mathbb{A},\partial\mathbb{B}]\!]_{n+\frac{1}{2}} &= \partial[\![\mathbb{A},\mathbb{B}]\!]_{n+\frac{1}{2}} + n[\![\mathbb{A},\mathbb{B}]\!]_{n-\frac{1}{2}}\ .
\end{aligned}
$$

Notice that the above relations imply that D (resp. ∂) is an odd (resp. even) derivation over all of the $[\![-,-]\!]_r$, for $2r \in \mathbb{Z}$; that is,

$$
\begin{aligned}
D[\![\mathbb{A},\mathbb{B}]\!]_r &= [\![D\mathbb{A},\mathbb{B}]\!]_r + (-)^{|\mathbb{A}|+2r}[\![\mathbb{A},D\mathbb{B}]\!]_r \\
\partial[\![\mathbb{A},\mathbb{B}]\!]_r &= [\![\partial\mathbb{A},\mathbb{B}]\!]_r + [\![\mathbb{A},\partial\mathbb{B}]\!]_r \ .
\end{aligned}
$$

The sign in the first equation can be understood if we notice that the $[\![-,-]\!]_r$ brackets for $r-\frac{1}{2}\in\mathbb{Z}$ have odd parity, so that the operation $\mathrm{ad}_r\mathbb{A} \equiv [\![\mathbb{A},-]\!]_r$ has parity $|\mathrm{ad}_r\mathbb{A}| = |\mathbb{A}| + 2r$.

Next the commutativity axiom takes the form (for $n \in \mathbb{Z}$):

$$
\begin{aligned}
[\![\mathbb{A},\mathbb{B}]\!]_n &= (-)^{|\mathbb{A}||\mathbb{B}|+n} \sum_{m\geq 0} \frac{(-)^m}{m!}\partial^m [\![\mathbb{B},\mathbb{A}]\!]_{n+m} \\
[\![\mathbb{A},\mathbb{B}]\!]_{n+\frac{1}{2}} &= (-)^{|\mathbb{A}||\mathbb{B}|+n} \sum_{m\geq 0} \frac{(-)^m}{m!}\partial^m \left([\![\mathbb{B},\mathbb{A}]\!]_{n+m+\frac{1}{2}} - D[\![\mathbb{B},\mathbb{A}]\!]_{n+m+1}\right) \ .
\end{aligned}
$$

In particular, the normal ordered product obeys:

$$
(\mathbb{A}\mathbb{B}) \equiv [\![\mathbb{A},\mathbb{B}]\!]_0 = (-)^{|\mathbb{A}||\mathbb{B}|} \sum_{m\geq 0} \frac{(-)^m}{m!}\partial^m [\![\mathbb{B},\mathbb{A}]\!]_m \ ,
$$

so that the normal ordered commutator is given by

$$
(\mathbb{A}\mathbb{B}) - (-)^{|\mathbb{A}||\mathbb{B}|}(\mathbb{B}\mathbb{A}) = (-)^{|\mathbb{A}||\mathbb{B}|} \sum_{m\geq 1} \frac{(-)^m}{m!}\partial^m [\![\mathbb{B},\mathbb{A}]\!]_m \ ,
$$

which coincides with the analogous formula in the nonsupersymmetric case.

Now we come to the associativity axioms. In the formulas which follow, $n,m \in \mathbb{Z}$ and in addition we take $m>0$ in the first and third and $m\geq 0$ in the remaining two:

$$
\begin{aligned}
[\![\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]_n]\!]_m &= (-)^{|\mathbb{A}||\mathbb{B}|}[\![\mathbb{B},[\![\mathbb{A},\mathbb{C}]\!]_m]\!]_n \\
&\quad + \sum_{q\geq 0}\binom{m-1}{q}[\![[\![\mathbb{A},\mathbb{B}]\!]_{q+1},\mathbb{C}]\!]_{m+n-q-1} \ , \\
[\![\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]_n]\!]_{m+\frac{1}{2}} &= (-)^{|\mathbb{B}|(|\mathbb{A}|+1)}[\![\mathbb{B},[\![\mathbb{A},\mathbb{C}]\!]_{m+\frac{1}{2}}]\!]_n \\
&\quad + \sum_{q\geq 0}\binom{m}{q}[\![[\![\mathbb{A},\mathbb{B}]\!]_{q+\frac{1}{2}},\mathbb{C}]\!]_{m+n-q} \ , \\
[\![\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]_{n+\frac{1}{2}}]\!]_m &= (-)^{|\mathbb{A}|(|\mathbb{B}|+1)}[\![\mathbb{B},[\![\mathbb{A},\mathbb{C}]\!]_m]\!]_{n+\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{q\geq 0} \binom{m-1}{q} \Big([\![[\![\mathbb{A},\mathbb{B}]\!]_{q+1},\mathbb{C}]\!]_{m+n-q-\frac{1}{2}} \\
&\qquad - [\![[\![\mathbb{A},\mathbb{B}]\!]_{q+\frac{1}{2}},\mathbb{C}]\!]_{m+n-q} \Big) \ , \\
[\![\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]_{n+\frac{1}{2}}]\!]_{m+\frac{1}{2}} &= (-)^{(|\mathbb{A}|+1)(|\mathbb{B}|+1)} [\![\mathbb{B},[\![\mathbb{A},\mathbb{C}]\!]_{m+\frac{1}{2}}]\!]_{n+\frac{1}{2}} \\
&+ (-)^{|\mathbb{A}|} \sum_{q\geq 0} \binom{m}{q} [\![[\![\mathbb{A},\mathbb{B}]\!]_{q+\frac{1}{2}},\mathbb{C}]\!]_{m+n-q+\frac{1}{2}} \\
&- (-)^{|\mathbb{A}|} \sum_{q\geq 0} m \binom{m-1}{q} [\![[\![\mathbb{A},\mathbb{B}]\!]_{q+1},\mathbb{C}]\!]_{m+n-q} \ .
\end{aligned}
$$

In particular, these axioms imply that the operation $[\![\mathbb{A},-]\!]_{\frac{1}{2}}$ is a derivation of parity $|\mathbb{A}|+1$ over all the $[\![-,-]\!]_r$:

$$
[\![\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]_r]\!]_{\frac{1}{2}} = (-)^{2r|\mathbb{A}|} [\![[\![\mathbb{A},\mathbb{B}]\!]_{\frac{1}{2}},\mathbb{C}]\!]_r + (-)^{(|\mathbb{A}|+1)(|\mathbb{B}|+2r)} [\![\mathbb{B},[\![\mathbb{A},\mathbb{C}]\!]_{\frac{1}{2}}]\!]_r \ .
$$

Moreover it follows that $[\![\mathbb{A},-]\!]_1$ is a derivation of parity $|\mathbb{A}|$ over the integral $[\![-,-]\!]_n$, but not over the half-integral $[\![-,-]\!]_{n+\frac{1}{2}}$. In particular, it is a derivation over the normal ordered product.

Finally, we arrive at the rearrangement lemma which are crucial in bringing normal ordered products to a standard form:

$$
(\mathbb{A}(\mathbb{B}\mathbb{C})) - (-)^{|\mathbb{A}||\mathbb{B}|}(\mathbb{B}(\mathbb{A}\mathbb{C})) = ((\mathbb{A}\mathbb{B})\mathbb{C}) - (-)^{|\mathbb{A}||\mathbb{B}|}((\mathbb{B}\mathbb{A})\mathbb{C}) \ .
$$

References

[1] E Abdalla and MCB Abdalla, *Supersymmetric extension of the chiral model and the Wess-Zumino term in two dimensions*, Phys. Lett. **152B** (1985) 59; P di Vecchia, VG Knizhnik, JL Petersen and P Rossi, *A supersymmetric Wess-Zumino lagrangian in two-dimensions*, Nucl. Phys. **B253** (1985) 701.

[2] K Bardakci and MB Halpern, Phys. Rev. **D3** (1971) 2493; MB Halpern, Phys. Rev. **D4** (1971) 2398

[3] JM Figueroa-O'Farrill, *Affine algebras, N=2 superconformal algebras, and gauged WZNW models*, `hep-th/9306164`, Phys. Lett. **316B** (1993) 496.

[4] JM Figueroa-O'Farrill and S Stanciu, *Nonsemisimple Sugawara constructions*, `hep-th/9402035`, Phys. Lett. **327B** (1994) 40.

[5] JM Figueroa-O'Farrill, *N=2 structures on solvable Lie algebras: the c=9 classification*, `hep-th/9412008`, Comm. Math. Phys. **177** (1996) 129.

[6] JM Figueroa-O'Farrill and S Stanciu, *Nonreductive WZW models and their CFTs*, `hep-th/9506151`, Nucl. Phys. **B458** (1996) 137-164.

[7] JM Figueroa-O'Farrill and S Stanciu, *On the structure of symmetric self-dual Lie algebras*, `hep-th/9506152`, J. Math. Phys. in print.

[8] JM Figueroa-O'Farrill and S Stanciu, *N=1 and N=2 cosets from gauged supersymmetric WZW models*, `hep-th/9511229`.

[9] E Getzler, *Manin triples and N=2 superconformal field theory*, `hep-th/9307041`.

[10] P Goddard, A Kent, and D Olive, *Virasoro algebras and coset space models*, Phys. Lett. **152B** (1985) 88, and *Unitary representations of the Virasoro and super Virasoro algebras*, Comm. Math. Phys. **103** (1986) 105.

[11] CM Hull and E Witten, *Supersymmetric sigma models and the heterotic string*, Phys. Lett. **160B** (1985) 398-402.

[12] VG Kač and I Todorov, *Superconformal current algebras and their unitary representations*, Comm. Math. Phys. **102** (1985) 337-347;
J Fuchs, *Superconformal Ward identities and the WZW model*, Nucl. Phys. **B286** (1986) 455.

[13] D Karabali, QH Park, HJ Schnitzer, and Z Yang, *A GKO construction based on a path integral formulation of gauged Wess-Zumino-Witten actions*, Phys. Lett. **216B** (1989) 307;
D Karabali and HJ Schnitzer, *BRST quantization of the gauged WZW action and coset conformal field theories*, Nucl. Phys. **B329** (1990) 649.

[14] Y Kazama and H Suzuki, *New N=2 superconformal field theories and superstring compactification*, Nucl. Phys. **B321** (1989) 232-268, and *Characterisation of N=2 superconformal models generated by coset space method*, Phys. Lett. **216B** (1989) 112.

[15] S Krivonos and K Thielemans, *A Mathematica Package for Computing N=2 Superfield Operator Product Expansions*, `hep-th/9512029`.

[16] N Mohammedi, *On bosonic and supersymmetric current algebras for nonsemisimple groups*, `hep-th/9312182`, Phys. Lett. **325B** (1994) 371-376.

[17] N Mohammedi, *N=2 current algebras for nonsemisimple groups*, `hep-th/9408048`, Phys. Lett. **337B** (1994) 279-284.

[18] T Nakatsu, *Supersymmetric gauged Wess–Zumino–Witten models*, Prog. Theor. Phys. **B87** (1992) 795–798.

[19] CR Nappi and E Witten, *A WZW model based on a non-semi-simple group*, `hep-th/9310112`, Phys. Rev. Lett. **71** (1993) 3751-3753.

[20] S Nojiri, *Superstring in two dimensional black hole*, Phys. Lett. **274B** (1992) 41–46.

[21] SE Parkhomenko, *Extended superconformal conformal algebras and finite-dimensional Manin triples*, Sov. Phys. JETP **B75** (1992) 1-3.

[22] A Polyakov and PB Wiegmann, *Theory of nonabelian Goldstone bosons in two dimensions*, Phys. Lett. **131B** (1983) 121; *Goldstone fields in two dimensions with multivalued actions*, Phys. Lett. **141B** (1984) 223.

[23] H Rhedin, *Gauged supersymmetric WZNW model using the BRST approach*, `hep-th/9511143`.

[24] H Schnitzer, *A path integral construction of superconformal field theories from a gauged supersymmetric Wess-Zumino-Witten action*, Nucl. Phys. **B324** (1989) 412-426.

[25] C Schweigert, *On the classification of $N=2$ superconformal coset theories*, Comm. Math. Phys. **149** (1992) 425.

[26] A Sevrin, Ph Spindel, W Troost and A Van Proeyen, *Extended supersymmetric σ-models on group manifolds (I): The complex structures*, Nucl. Phys. **B308** (1988) 662-698; and *(II): Current algebras*, Nucl. Phys. **B311** (1988/89) 465-492.

[27] A Tseytlin, *Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories*, Nucl. Phys. **B411** (1994) 509-558.

[28] E Witten, *Nonabelian bosonization in two-dimensions*, Comm. Math. Phys. **92** (1984) 455-472.

[29] E Witten, *The N matrix model and gauged WZW models*, Nucl. Phys. **B371** (1992) 191-245.